

# Isomorphism Conjecture for homotopy $K$ -theory and groups acting on trees

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## Abstract

We discuss an analogon to the Farrell-Jones Conjecture for homotopy algebraic  $K$ -theory. In particular, we prove that if a group  $G$  acts on a tree and all isotropy groups satisfy this conjecture, then  $G$  satisfies this conjecture. This result can be used to get rational injectivity results for the assembly map in the Farrell-Jones Conjecture in algebraic  $K$ -theory.

Key words:  $K$ -theory and homotopy  $K$ -theory of group rings, Isomorphism Conjectures, Actions on trees.

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## 0 Introduction

The Farrell-Jones Conjecture [12] in algebraic  $K$ -theory is concerned with the  $K$ -theory  $K_n(RG)$  of group rings  $RG$  for a group  $G$  and a ring  $R$ . The conjecture states that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R) \rightarrow K_n(RG) \quad (0.1)$$

is an isomorphism. (This map is constructed by applying a certain  $G$ -homology theory  $H_n^G(-; \mathbf{K}_R)$  to the projection  $E_{\mathcal{VCYC}}(G) \rightarrow \text{pt}$ , see Definition 1.1 and Remark 6.6.) There seem to occur two quite different phenomena in the algebraic  $K$ -theory of such group rings. Firstly,  $K_n(RG)$

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contains elements coming from the  $K$ -theory of  $RF$  for finite subgroups  $F$  of  $G$ . Secondly, it contains nilgroup information. This is already illuminated in the simple case  $G = \mathbb{Z}$ , then  $R[\mathbb{Z}] = R[t, t^{-1}]$  and by the Bass-Heller-Swan formula [7], [14]

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \quad (0.2)$$

Here  $NK_n(R)$  are the Nil-groups of  $R$ , which can be defined as the kernel of the projection  $K_n(R[t]) \rightarrow K_n(R)$  induced from  $t \mapsto 0$ . In general it is known [1] that the domain of the assembly map (0.1) splits as

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VCYC}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_R). \quad (0.3)$$

Thus, the Farrell-Jones Conjecture predicts a similar splitting for  $K_n(RG)$ .

In this paper we will formulate a (Fibered) Isomorphism Conjecture for homotopy algebraic  $K$ -theory, see Conjecture 7.3. This variant of  $K$ -theory was defined by Weibel [31], building on the definition of Karoubi-Villamayor  $K$ -theory. The homotopy algebraic  $K$ -theory groups of a ring  $R$  are denoted by  $KH_n(R)$ . Their crucial property is homotopy invariance:  $KH_n(R) \cong KH_n(R[t])$ . In particular, homotopy algebraic  $K$ -theory does not contain Nil-groups. We think about this  $KH$ -Isomorphism Conjecture as an Isomorphism Conjecture for algebraic  $K$ -theory *modulo Nil-groups*. For a more precise formulation of the relation of the Farrell-Jones Conjecture in algebraic  $K$ -theory to the  $KH$ -Isomorphism Conjecture see Section 8.

Our main results concerning the  $KH$ -Isomorphism Conjecture are inheritance properties. A group  $G$  acts on a tree  $T$ , if  $T$  is a 1-dimensional  $G$ -CW-complex which is contractible (after forgetting the group action).

**Definition 0.4 (The class of groups  $\mathcal{C}_0$ ).** We define the following properties a class  $\mathcal{C}$  of groups may or may not have.

- (FIN) All finite groups belong to  $\mathcal{C}$ ;
- (TREE) Suppose that  $G$  acts on a tree  $T$ . Assume that for each  $x \in T$  the isotropy group  $G_x$  belongs to  $\mathcal{C}$ . Then  $G$  belongs to  $\mathcal{C}$ ;
- (COL) Let  $G$  be a group with a directed system of subgroups  $\{G_i \mid i \in I\}$ , which is directed by inclusion and satisfies  $\bigcup_{i \in I} G_i = G$ . If each  $G_i$  belongs to  $\mathcal{C}$ , then  $G \in \mathcal{C}$ ;
- (SUB) If  $G \in \mathcal{C}$  and  $H \subseteq G$  is a subgroup, then  $H \in \mathcal{C}$ .

We define  $\mathcal{C}_0$  to be the smallest class of groups satisfying (FIN), (TREE) and (COL).

It is not hard to check that the class  $\mathcal{C}_0$  is closed under taking subgroups. For instance if  $H$  is a subgroup of a group  $G$  acting on a tree, then  $H$  acts also on this tree and the isotropy groups satisfy  $H_x \subseteq G_x$ . By induction we may assume that the  $G_x$  are closed under taking subgroups and therefore  $H \in \mathcal{C}_0$ .

**Theorem 0.5. (Inheritance properties of the  $KH$ -Isomorphism Conjecture)** *The class of groups satisfying the Fibered  $KH$ -Isomorphism Conjecture for a fixed coefficient ring  $R$  has the properties (FIN), (TREE), (COL) and (SUB). The class of groups satisfying the  $KH$ -Isomorphism Conjecture for a fixed coefficient ring  $R$  has the properties (FIN), (TREE) and (COL). In particular, all groups in  $\mathcal{C}_0$  satisfy the (Fibered)  $KH$ -Isomorphism Conjecture.*

**Remark 0.6.** The class of groups satisfying the  $KH$ -Isomorphism Conjecture is strictly bigger than  $\mathcal{C}_0$  since it contains all fundamental groups of closed Riemannian manifolds with negative sectional curvature by [5] and Theorem 8.4 (i).

This result has the following applications.

**Theorem 0.7 (Extensions of groups and actions on trees).** *Let  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  be an extension of groups. Suppose that  $K$  acts on a tree with finite stabilizers and that  $Q$  satisfies the Fibered  $KH$ -Isomorphism Conjecture 7.3 for the ring  $R$ . Then  $G$  satisfies the Fibered  $KH$ -Isomorphism Conjecture 7.3 for the ring  $R$ .*

A ring  $R$  is called *regular* if it is Noetherian and every finitely generated  $R$ -module possesses a finite-dimensional resolution by finitely generated projective modules.

**Theorem 0.8. (Conclusions for the  $K$ -theoretic Farrell-Jones Conjecture for groups in  $\mathcal{C}$ ).** *Let  $G$  be a group in the class  $\mathcal{C}_0$  defined above in (0.4). Then*

(i) *Let  $R$  be a regular ring with  $\mathbb{Q} \subseteq R$ . Then the assembly map*

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

*is injective, or, equivalently, the injectivity part of the Farrell-Jones Isomorphism Conjecture for algebraic  $K$ -theory is true for  $(G, R)$ .*

(ii) *Let  $R$  be the ring  $\mathbb{Z}$  of integers. Then the assembly map*

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \rightarrow K_n(\mathbb{Z}G)$$

*is rationally injective, or, equivalently, the rational injectivity part of the Farrell-Jones Isomorphism Conjecture for algebraic  $K$ -theory is true for  $(G, \mathbb{Z})$ .*

**Proposition 0.9.** *The following classes of groups belong to  $\mathcal{C}_0$ :*

(i) *One relator groups;*

(ii)  $G$  is poly-free, i.e. there is a filtration

$$\{1\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots \subseteq G_n = G$$

such that  $G_i$  is normal in  $G_{i+1}$  with a free group as quotient  $G_{i+1}/G_i$ .  
The pure braid group is an example;

(iii) Let  $M$  be a compact orientable 3-manifold with prime decomposition  $M = M_1 \# M_2 \# \dots \# M_n$ . Suppose that each  $M_i$ , which has infinite fundamental group and is aspherical, has a boundary or is a Haken manifold. Then  $\pi_1(M) \in \mathcal{C}_0$ ;

(iv) If  $M$  is a compact 2-dimensional manifold, then  $\pi_1(M) \in \mathcal{C}_0$ ;

(v) If  $M$  is a submanifold of  $S^3$ , then  $\pi_1(M) \in \mathcal{C}_0$ .

Next we discuss similar inheritance properties for the Farrell-Jones Conjecture in algebraic K-theory. A ring  $R$  is called *regular coherent* if every finitely presented  $R$ -module possesses a finite-dimensional resolution by finitely generated projective  $R$ -modules. A ring  $R$  is regular if and only if it is regular coherent and Noetherian. A group  $G$  is called *regular* or *regular coherent* respectively if for any regular ring  $R$  the group ring  $RG$  is regular respectively regular coherent. For more information about these notions we refer to [29, Theorem 19.1].

**Definition 0.10 (The classes of groups  $\mathcal{C}$  and  $\mathcal{C}'$ ).** Consider the following further properties a class  $\mathcal{C}$  of groups may have.

**(TRI)** The trivial group belongs to  $\mathcal{C}$ ;

**(VCYC)** All virtually cyclic groups belong to  $\mathcal{C}$ ;

**(TREE $_{\mathcal{R}}$ )** Suppose that  $G$  acts on a tree  $T$ . Assume that for each  $x \in T$  the isotropy group  $G_x$  belongs to  $\mathcal{C}$ . For each edge  $e$  of  $T$ , assume that the isotropy group  $G_e$  is regular coherent. Then  $G$  belongs to  $\mathcal{C}$ ;

The class  $\mathcal{C}$  is defined as the smallest class of groups satisfying (TRI), (TREE $_{\mathcal{R}}$ ) and (COL). The class  $\mathcal{C}'$  is defined as the smallest class of groups satisfying (VCYC), (TREE $_{\mathcal{R}}$ ) and (COL).

All groups appearing in  $\mathcal{C}$  are torsionfree. Similar to the class  $\mathcal{C}_0$ , the classes  $\mathcal{C}$  and  $\mathcal{C}'$  are closed under taking subgroups [29, Proposition 19.3]. We conclude from Waldhausen [29, Theorem 17.5 on page 250] that  $\mathcal{C}$  contains a group  $G$  appearing in Proposition 0.9 under (i), (iii), (iv), and (v) provided that  $G$  is torsionfree. One of the main results in Waldhausen's article [29] is that for a regular ring  $R$  the K-theoretic assembly map

$$H_n(BG; \mathbf{K}_R) \rightarrow K_n(RG)$$

is an isomorphism. Actually, Waldhausen states this only for  $n \geq 0$ , but the embedding of  $K_{n-1}(R)$  into  $K_n(R[\mathbb{Z}])$  allows the extension to all  $n$ , see for example Remark 10.3. Furthermore, Waldhausen considers HNN-extensions and amalgamated products rather than action on trees, but this does not change the class  $\mathcal{CL}$ , compare Remark 4.7 and Lemma 5.1.

**Theorem 0.11.** *Let  $R$  be a regular ring. The class of groups satisfying the Farrell-Jones Conjecture in algebraic  $K$ -theory for the ring  $R$  has the properties (VCYC),  $(TREE_R)$  and (COL). In particular, all groups in  $\mathcal{CL}'$  satisfy the Farrell-Jones Conjecture in algebraic  $K$ -theory for the ring  $R$ .*

Related results can be found in [15] and [24]. It is an interesting question, whether the class of groups satisfying the Farrell-Jones Conjecture in algebraic  $K$ -theory has the property (TREE).

Theorem 0.5 and Waldhausen's result imply

**Corollary 0.12.** *Let  $R$  be a regular ring. Let  $G$  be a group in the class  $\mathcal{CL}$ . Then the canonical map*

$$K_n(RG) \rightarrow KH_n(RG)$$

*is bijective for  $n \in \mathbb{Z}$ .*

Isomorphism Conjectures can be formulated in the quite general context of equivariant homology theories, see Definition 1.1. We show in Theorem 4.2 that the property (TREE) holds for the class of groups satisfying the Isomorphism Conjecture for such an equivariant homology theory whenever the equivariant homology theory satisfies the *tree property*, see Definitions 4.1. The weaker property  $(TREE_R)$  is related to the *regular tree property* (see Definition 4.1), which is a weakening of the tree property. The above Theorem 0.7 has also an analogon in this setting, see Corollary 4.4. The tree property means essentially that there are Mayer-Vietoris sequences for amalgamated products and HNN extensions of groups in the equivariant homology theory (see Lemma 4.5). On spectrum level this means that there are certain homotopy cartesian diagrams of spectra (see Lemma 5.1 and Remark 5.6). In Section 7 we define the equivariant homology theory  $H_*^?(-, \mathbf{KH}_R)$  that is relevant for the  $KH$ -Isomorphism Conjecture. We prove in Theorem 11.1 that this theory satisfies the tree property. In the case of algebraic  $K$ -theory amalgamated free products and HNN extensions have been analyzed by Waldhausen [29]. In both cases there are long exact sequences, but they involve as an additional term Waldhausen's Nil-groups. Their nontriviality obstructs the equivariant homology theory  $H_*^?(-; \mathbf{K}_R)$  relevant for the Farrell-Jones Conjecture in algebraic  $K$ -theory from having the tree property. Our proof of the tree property for  $H_*^?(-, \mathbf{KH}_R)$ , consists essentially of showing that Waldhausen's Nil-groups are killed under the transition from  $K$  to  $KH$ . On the other hand vanishing results for Waldhausen's Nil-groups can be used to show (see Theorem 11.4) that  $H_*^?(-; \mathbf{K}_R)$  has the *regular tree property*

(see Definition 4.1). This implies then, that for a regular ring  $R$  the class of groups satisfying the Farrell-Jones Conjecture has the property  $(\text{TREE}_{\mathcal{R}})$ , see Theorem 4.2 (iii). It is an interesting question for which rings  $R$  the equivariant homology theory  $H_*^?(-; \mathbf{K}_R) \otimes \mathbb{Q}$  has the tree property. It is worthwhile to consider also  $H_*^?(-; \mathbf{L}_R^{-\infty})$ , the equivariant homology theory relevant for the Farrell-Jones Conjecture in  $L$ -theory. In this case amalgamated free products and HNN extensions have been analyzed by Cappell [9]. Again additional terms appear in the long exact sequences, the UNil-groups and non-triviality of those obstructs this theory from having the tree property. On the other hand these UNil-groups are known to be 2-torsion [9], thus  $H_n^?(-; \mathbf{L}_R^{-\infty}) \otimes \mathbb{Z}[\frac{1}{2}]$  does have the tree property. Thus we obtain the following result.

**Theorem 0.13. (Conclusions for the  $L$ -theoretic Farrell-Jones Conjecture for groups in  $\mathcal{C}$ ).** *The class of groups for which the assembly map*

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{-\infty}) \rightarrow L_n^{-\infty}(RG)$$

*becomes an isomorphism after tensoring with  $\mathbb{Z}[\frac{1}{2}]$ , has the properties (FIN), (TREE) and (COL). In particular, this class contains the class in  $\mathcal{C}_0$  from Definition 0.4.*

In the context of topological  $K$ -theory, i.e. for the Baum-Connes Conjecture one can apply our results to the equivariant  $K$ -theory  $H_n^G(-; \mathbf{K}^{\text{top}}) = K_n^G(-)$ . Then one obtains the analogon of our Theorem 0.5. In this case amalgamated products and HNN extensions have been analyzed Pimsner-Voiculescu [20] and Pimsner [21]. Here the situation is much better, since no Nil-groups appear. This analogon has already been proved by Oyono-Oyono [19] for the Baum-Connes Conjecture (with coefficients).

We are indebted to Holger Reich for pointing out the reference [30] to us.

The papers is organized as follows:

1. Isomorphism Conjectures for equivariant homology theories
  2. Homological aspects
  3. Continuous equivariant homology theories
  4. The tree property
  5. Equivariant homology theories constructed from spectra
  6. Isomorphism Conjectures for spectra
  7. The  $KH$ -Isomorphism Conjecture
  8. The Relation between the  $K$ - and the  $KH$ -Isomorphism Conjecture
  9. Non-connective Waldhausen Nil
  10. Waldhausen's cartesian squares
  11. The tree property for  $KH$
- References

# 1 Isomorphism Conjectures for equivariant homology theories

We will use the notion of an equivariant homology theory  $\mathcal{H}_*^?$  with values in  $\Lambda$ -modules for a commutative associative ring  $\Lambda$  with unit from [16, Section 1]. This essentially means that we get for each group  $G$  a  $G$ -homology theory  $\mathcal{H}_*^G$  which assigns to a (not necessarily proper or cocompact) pair of  $G$ -CW-complexes  $(X, A)$  a  $\mathbb{Z}$ -graded  $\Lambda$ -module  $\mathcal{H}_n^G(X, A)$ , such that there exists natural long exact sequences of pairs and  $G$ -homotopy invariance, excision, and the disjoint union axiom are satisfied. Moreover, an induction structure is required which in particular implies for a subgroup  $H \subseteq G$  and a  $H$ -CW-pair  $(X, A)$  that there is a natural isomorphism  $\mathcal{H}_n^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(G \times_H (X, A))$ .

We will later discuss examples, the most important ones will be given by those equivariant homology theories which appear in the Baum-Connes Conjecture and the Farrell-Jones Conjecture. These conjectures are special cases of the following more general formulation of a (Fibered) Isomorphism Conjecture (see Section 6).

A family  $\mathcal{F}$  of subgroups of  $G$  is a set of subgroups which is closed under conjugation and taking subgroups. If  $\mathcal{C}$  is a class of groups that is closed under taking subgroups and isomorphisms, then the collections of subgroups of  $G$  that are in  $\mathcal{C}$  forms a family  $\mathcal{C}(G)$  of subgroups of  $G$ . Abusing notation, we will denote this family often by  $\mathcal{C}$ . Examples are the families  $\mathcal{FIN}$  of finite subgroups and  $\mathcal{VCYC}$  of virtually cyclic subgroups. Given a group homomorphism  $\phi: K \rightarrow G$  and a family  $\mathcal{F}$  of subgroups of  $G$ , define the family  $\phi^*\mathcal{F}$  of subgroups of  $K$  by  $\phi^*\mathcal{F} = \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}$ . If  $i: H \rightarrow G$  is the inclusion of a subgroup, then we write often  $\mathcal{F}|_H$  for  $i^*\mathcal{F}$ . Associated to such a family there is a  $G$ -CW-complex  $E_{\mathcal{F}}(G)$  (unique up to  $G$ -homotopy equivalence) with the property that the fixpoint sets  $E_{\mathcal{F}}(G)^H$  are contractible for  $H \in \mathcal{F}$  and empty for  $H \notin \mathcal{F}$ . For  $\mathcal{F} = \mathcal{ALL}$  the family of all subgroups, we can take the one-point-space  $\text{pt}$  as a model for  $E_{\mathcal{ALL}}(G)$ . For more information about the spaces we refer for instance to [17].

**Definition 1.1 ((Fibered) Isomorphism Conjecture for  $\mathcal{H}_*^?$ ).** Let  $\mathcal{H}_*^?$  be an equivariant homology theory with values in  $\Lambda$ -modules. A group  $G$  together with a family of subgroups  $\mathcal{F}$  satisfies the *Isomorphism Conjecture (in the range  $\leq N$ )* if the projection  $\text{pr}: E_{\mathcal{F}}(G) \rightarrow \text{pt}$  to the one-point-space  $\text{pt}$  induces an isomorphism

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathcal{H}_n^G(\text{pt})$$

for  $n \in \mathbb{Z}$  (with  $n \leq N$ ).

The pair  $(G, \mathcal{F})$  satisfies the *Fibered Isomorphism Conjecture (in the range  $\leq N$ )* if for each group homomorphism  $\phi: K \rightarrow G$  the pair  $(K, \phi^*\mathcal{F})$  satisfies the Isomorphism Conjecture (in the range  $\leq N$ ).

Built in into the Fibered Isomorphism Conjecture is the following obvious inheritance property which is not known to be true in general in the non-fibered case.

**Lemma 1.2.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory, let  $\phi: K \rightarrow G$  be a group homomorphism and let  $\mathcal{F}$  be a family of subgroups. If  $(G, \mathcal{F})$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ), then  $(K, \phi^*\mathcal{F})$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

*Proof.* If  $\psi: L \rightarrow K$  is a group homomorphism, then  $\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}$ .  $\square$

In particular, if for a given class of groups  $\mathcal{C}$ , which is closed under isomorphism and taking subgroups, the Fibered Isomorphism Conjecture 1.1 is true for  $(G, \mathcal{C}(G))$  and if  $H \subseteq G$  is a subgroup, then the Fibered Isomorphism Conjecture 1.1 is true for  $(H, \mathcal{C}(H))$ .

## 2 Homological aspects

The disjoint union axiom ensures that a  $G$ -homology is compatible with directed colimits.

**Lemma 2.1.** *Let  $\mathcal{H}_*^G$  be a  $G$ -homology theory. Let  $X$  be a  $G$ -CW-complex and  $\{X_i \mid i \in I\}$  be a directed system of  $G$ -CW-subcomplexes directed by inclusion such that  $X = \bigcup_{i \in I} X_i$ . Then for all  $n \in \mathbb{Z}$  the natural map*

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

*is bijective.*

*Proof.* Compare for example with [27, Proposition 7.53 on page 121], where the non-equivariant case for  $I = \mathbb{N}$  is treated. The main point is that the functor colimit over a directed system of  $R$ -modules is exact.  $\square$

**Lemma 2.2.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory with values in  $\Lambda$ -modules in the sense of [16, Section 1]. Let  $G$  be a group and let  $\mathcal{F}$  be a family of subgroups of  $G$ . Let  $Z$  be a  $G$ -CW-complex. Consider  $N \in \mathbb{Z} \cup \{\infty\}$ . Suppose for each  $H \subseteq G$  which occurs as isotropy group in  $Z$  that the  $G$ -map induced by the projection  $\operatorname{pr}: E_{\mathcal{F}|_H}(H) \rightarrow \operatorname{pt}$*

$$\mathcal{H}_n^H(\operatorname{pr}): \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \rightarrow \mathcal{H}_n^H(\operatorname{pt})$$

*is bijective for all  $n \in \mathbb{Z}, n \leq N$ .*

*Then the map induced by the projection  $\operatorname{pr}_1: E_{\mathcal{F}}(G) \times Z \rightarrow Z$*

$$\mathcal{H}_n^G(\operatorname{pr}_1): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \rightarrow \mathcal{H}_n^G(Z)$$

*is bijective for  $n \in \mathbb{Z}, n \leq N$ .*



*Proof.* We first prove the claim for finite-dimensional  $G$ -CW-complexes by induction over  $d = \dim(Z)$ . The induction beginning  $\dim(Z) = -1$ , i.e.  $Z = \emptyset$ , is trivial. In the induction step from  $(d-1)$  to  $d$  we choose a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d & \longrightarrow & Z_d \end{array}$$

If we cross it with  $E_{\mathcal{F}}(G)$ , we obtain another  $G$ -pushout of  $G$ -CW-complexes. The various projections induce a map from the Mayer-Vietoris sequence of the latter  $G$ -pushout to the Mayer-Vietoris sequence of the first  $G$ -pushout. By the Five-Lemma it suffices to prove that the following maps

$$\begin{aligned} \mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G \left( E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times S^{d-1} \right) &\rightarrow \mathcal{H}_n^G \left( \coprod_{i \in I_d} G/H_i \times S^{d-1} \right); \\ \mathcal{H}_n^G(\text{pr}_3): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) &\rightarrow \mathcal{H}_n^G(Z_{d-1}); \\ \mathcal{H}_n^G(\text{pr}_4): \mathcal{H}_n^G \left( E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times D^n \right) &\rightarrow \mathcal{H}_n^G \left( \coprod_{i \in I_d} G/H_i \times D^n \right), \end{aligned}$$

are bijective for  $n \in \mathbb{Z}, n \leq N$ . This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and  $G$ -homotopy invariance of  $\mathcal{H}_*^G$  the claim follows for the third map if we can show for any  $H \subseteq G$  which occurs as isotropy group in  $Z$  that the map

$$\mathcal{H}_n^G(\text{pr}_1): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \rightarrow \mathcal{H}_n^G(G/H) \quad (2.3)$$

is bijective for  $n \in \mathbb{Z}, n \leq N$ . The  $G$ -map

$$G \times_H \text{res}_G^H E_{\mathcal{F}}(G) \rightarrow G/H \times E_{\mathcal{F}}(G) \quad (g, x) \mapsto (gH, gx)$$

is a  $G$ -homeomorphism where  $\text{res}_G^H$  denotes the restriction of the  $G$ -action to an  $H$ -action. Obviously  $\text{res}_G^H E_{\mathcal{F}}(G)$  is a model for  $E_{\mathcal{F}|_H}(H)$ . Since for any  $H$ -CW-complex  $Y$  there is a natural isomorphism  $\mathcal{H}_n^H(Y) \xrightarrow{\cong} \mathcal{H}_n^G(G \times_H Y)$ , the map (2.3) can be identified with the map

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt})$$

which is bijective for all  $n \in \mathbb{Z}, n \leq N$  by assumption. This finishes the proof in the case that  $Z$  is finite-dimensional.

Finally we consider an arbitrary  $G$ -CW-complex  $Z$ . It can be written as the colimit  $\text{colim}_{d \rightarrow \infty} Z_d$ . The natural maps

$$\begin{aligned} \text{colim}_{d \rightarrow \infty} \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_d) &\xrightarrow{\cong} \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z); \\ \text{colim}_{d \rightarrow \infty} \mathcal{H}_n^G(Z_d) &\xrightarrow{\cong} \mathcal{H}_n^G(Z), \end{aligned}$$

are bijective by Lemma 2.1. Since the colimit of isomorphisms is an isomorphism again, Lemma 2.2 follows.  $\square$

**Theorem 2.4 (Reducing the family).** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory with values in  $\Lambda$ -modules. Let  $G$  be a group and let  $\mathcal{F} \subseteq \mathcal{G}$  be families of subgroups of  $G$ . Consider  $N \in \mathbb{Z} \cup \{\infty\}$ . Suppose for each  $H \in \mathcal{G}$ , or, more generally, suppose for each isotropy group appearing in a specific model for  $E_{\mathcal{G}}(G)$  that  $(H, \mathcal{F}|_H)$  satisfies the (Fibered) Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

*Then  $(G, \mathcal{G})$  satisfies the (Fibered) Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) if and only if  $(G, \mathcal{F})$  satisfies the (Fibered) Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

*Proof.* For the Isomorphism Conjecture this follows from Lemma 2.2 applied to the case  $Z = E_{\mathcal{G}}(G)$  and the fact that  $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G)$  is a model for  $E_{\mathcal{F}}(G)$ . The case of the Fibered Isomorphism Conjecture is easily reduced to the former case.  $\square$

**Lemma 2.5.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory with values in  $\Lambda$ -modules. Let  $\mathcal{C}$  be a class of groups that is closed under isomorphisms, subgroups and quotients. Let  $1 \rightarrow L \rightarrow G \xrightarrow{p} Q \rightarrow 1$  be an extension of groups. Suppose that  $(Q; \mathcal{C}(Q))$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) and that for  $H \in p^*\mathcal{C}(Q)$  the pair  $(H, \mathcal{C}(H))$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

*Then  $(G, \mathcal{C}(G))$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

*Proof.* By Lemma 1.2 the pair  $(G, p^*\mathcal{C}(Q))$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ). Since  $\mathcal{C}$  is closed under quotients we have  $\mathcal{C}(G) \subseteq p^*\mathcal{C}(Q)$ . Now the assumption on the subgroups  $H \in p^*\mathcal{C}(Q)$  and Theorem 2.4 imply the result.  $\square$

### 3 Continuous equivariant homology theories

In this section we explain a criterion for an equivariant homology theory ensuring that for a class  $\mathcal{C}$  of groups closed under subgroups and isomorphisms the (Fibered) Isomorphism Conjecture 1.1 is true for  $(G, \mathcal{C}(G))$  provided that  $G$  is a directed union  $G = \bigcup_{i \in I} G_i$  of groups  $G_i$  and the (Fibered)  $\mathcal{C}$ -Isomorphism Conjecture 1.1 is true for  $(G_i, \mathcal{C}(G_i))$  for all  $i \in I$ .

**Definition 3.1 (Continuous equivariant homology theory).** An equivariant homology theory  $\mathcal{H}_*^?$  is called *continuous* if for each group  $G$  and directed system of subgroups  $\{G_i \mid i \in I\}$ , which is directed by inclusion and satisfies  $\bigcup_{i \in I} G_i = G$ , and each  $n \in \mathbb{Z}$  the map

$$\text{colim}_{i \in I} j_i \colon \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{pt}) \rightarrow \mathcal{H}_n^G(\text{pt})$$

is an isomorphism, where  $j_i: \mathcal{H}_n^{G_i}(\text{pt}) \rightarrow \mathcal{H}_n^G(\text{pt})$  is the composition of the induction isomorphism  $\mathcal{H}_n^{G_i}(\text{pt}) \xrightarrow{\cong} \mathcal{H}_n^G(G/G_i)$  with the map induced by the projection  $G/G_i \rightarrow \text{pt}$ .

**Lemma 3.2.** *Let  $\mathcal{H}_*^?$  be a continuous equivariant homology theory. Let  $G$  be a group with a directed system of subgroups  $\{G_i \mid i \in I\}$ , which is directed by inclusion and satisfies  $\bigcup_{i \in I} G_i = G$ .*

*Then for each  $G$ -CW-complex  $X$  and each  $n \in \mathbb{Z}$  the map*

$$\text{colim}_{i \in I} j_i: \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} X) \rightarrow \mathcal{H}_n^G(X)$$

*is an isomorphism, where  $j_i: \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} X) \rightarrow \mathcal{H}_n^G(X)$  is the composition of the induction isomorphism  $\mathcal{H}_n^{G_i}(\text{res}_G^{G_i} X) \xrightarrow{\cong} \mathcal{H}_n^G(G \times_{G_i} \text{res}_G^{G_i} X)$  with the homomorphism induced by the  $G$ -map  $G \times_{G_i} \text{res}_G^{G_i} X \rightarrow X$  that sends  $(g, x)$  to  $gx$ .*

*Proof.* Since  $\text{colim}_{i \in I}$  is an exact functor,  $\text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} X)$  is a  $G$ -homology theory in  $X$ . The map  $\text{colim}_{i \in I} j_i$  is a transformation of  $G$ -homology theories. Therefore it suffices to prove that

$$\text{colim}_{i \in I} j_i: \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H) \rightarrow \mathcal{H}_n^G(G/H)$$

is an isomorphism for every subgroup  $H \subseteq G$  and  $n \in \mathbb{Z}$ .

For  $i \in I$  let  $k_i: G_i/G_i \cap H \rightarrow \text{res}_G^{G_i} G/H$  be the obvious injective  $G_i$  map. Then the following diagram commutes

$$\begin{array}{ccc} \text{colim}_{i \in I} \mathcal{H}_n^{G_i \cap H}(\text{pt}) & \xrightarrow{\cong} & \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(G_i/G_i \cap H) \\ \downarrow \text{colim}_{i \in I} j_i \cong & & \downarrow \text{colim}_{i \in I} k_i \\ & & \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H) \\ & & \downarrow \text{colim}_{i \in I} j_i \\ \mathcal{H}_n^H(\text{pt}) & \xrightarrow{\cong} & \mathcal{H}_n^G(G/H) \end{array}$$

where the horizontal maps are the isomorphism given by induction. The left vertical arrow is bijective since  $\mathcal{H}_*^?$  is continuous by assumption. Hence it remains to show that the map

$$\text{colim}_{i \in I} k_i: \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(G_i/G_i \cap H) \rightarrow \text{colim}_{i \in I} \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H) \quad (3.3)$$

is surjective.

We get an obvious decomposition of  $G_i$ -sets

$$\text{res}_G^{G_i} G/H = \coprod_{G_i gH \in G_i \backslash G/H} G_i/G_i \cap gHg^{-1}.$$

It induces an identification

$$\mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H) = \bigoplus_{G_i gH \in G_i \backslash G/H} \mathcal{H}_n^{G_i}(G_i/G_i \cap gHg^{-1}).$$

The summand corresponding to  $G_i 1H$  is precisely the image of

$$\mathcal{H}_n^{G_i}(k_i): \mathcal{H}_n^{G_i}(G_i/G_i \cap H) \rightarrow \mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H).$$

Consider an element  $G_i gH \in G_i \backslash G/H$ . Choose an index  $j$  with  $j \geq i$  and  $g \in G_j$ . Then the structure map for  $i \leq j$  is a map  $\mathcal{H}_n^{G_i}(\text{res}_G^{G_i} G/H) \rightarrow \mathcal{H}_n^{G_j}(\text{res}_G^{G_j} G/H)$  which sends the summand corresponding to  $G_i gH \in G_i \backslash G/H$  to the image of

$$\mathcal{H}_n^{G_j}(k_j): \mathcal{H}_n^{G_j}(G_j/G_j \cap H) \rightarrow \mathcal{H}_n^{G_j}(\text{res}_G^{G_j} G/H).$$

This implies that the map (3.3) is surjective. This finishes the proof of Lemma 3.2.  $\square$

**Proposition 3.4.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory which is continuous. Let  $\mathcal{C}$  be a class of groups that is closed under isomorphism and taking subgroups. Let  $G$  be the directed union  $G = \bigcup_{i \in I} G_i$  of subgroups  $G_i$  such that the (Fibered) Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) is true for  $(G_i, \mathcal{C}(G_i))$  for all  $i \in I$ .*

*Then the (Fibered) Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) is true for  $(G, \mathcal{C}(G))$ .*

*Proof.* Since  $\mathcal{H}_*^?$  is continuous by assumption, we get the isomorphism

$$\text{colim}_{i \in I} \mathcal{H}_*^{G_i}(\text{pt}) = \mathcal{H}_*^G(\text{pt})$$

and from Lemma 3.2 the isomorphism

$$\mathcal{H}_*^G(E_{\mathcal{C}(G)}(G)) = \text{colim}_{i \in I} \mathcal{H}_*^{G_i}(\text{res}_G^{G_i} E_{\mathcal{C}(G)}(G)).$$

The result follows for the Isomorphism Conjecture since the colimit of an isomorphism is an isomorphism and since  $\text{res}_G^{G_i} E_{\mathcal{C}(G)}(G)$  is a model for  $E_{\mathcal{C}(G_i)}(G_i)$ . If  $\phi: K \rightarrow G$  is a group homomorphism then the same argument can be applied to the triple  $(K, \phi^* \mathcal{C}(G), \{\phi^{-1}(G_i) \mid i \in I\})$  in place of  $(G, \mathcal{C}(G), \{G_i \mid i \in I\})$  and this implies the statement for the Fibered Isomorphism Conjecture.  $\square$

## 4 The tree property

In this section we study criteria for an equivariant homology theory that ensure that the class of groups  $G$  for which  $(G, \mathcal{FIN})$  satisfies the (Fibered) Isomorphism Conjecture 1.1 has property (TREE) from Definition 0.4 or that the class of groups  $G$  for which  $(G, \mathcal{VCC})$  satisfies the Isomorphism Conjecture 1.1 has property (TREE $_{\mathcal{R}}$ ) from Definition 0.10.

**Definition 4.1 (Tree Property).** An equivariant homology theory  $\mathcal{H}_*^?$  has the *tree property* if for any group  $G$  that acts on a tree  $T$ , the projection  $\text{pr}: T \rightarrow \text{pt}$  induces for all  $n \in \mathbb{Z}$  (with  $n \leq N$ ) isomorphisms

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^G(T) \rightarrow \mathcal{H}_n^G(\text{pt}).$$

It has the *regular tree property* if for any group  $G$  that acts on a tree  $T$ , such that for each edge  $e$  of  $T$  the isotropy group  $G_e$  is regular coherent, the projection  $\text{pr}: T \rightarrow \text{pt}$  induces for all  $n \in \mathbb{Z}$  (with  $n \leq N$ ) isomorphisms

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^G(T) \rightarrow \mathcal{H}_n^G(\text{pt}).$$

**Theorem 4.2. (The tree property and inheritance properties of Isomorphism Conjectures)** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory. Let  $\mathcal{C}$  be a class of groups closed under subgroups and isomorphisms. Let  $\mathcal{D}_{\text{fib}}(\mathcal{C})$  be the class of groups  $G$  for which Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) is true for  $(G, \mathcal{C}(G))$  and let  $\mathcal{D}(\mathcal{C})$  be the class of groups  $G$  for which Isomorphism Conjecture 1.1 (in the range  $\leq N$ ) is true for  $(G, \mathcal{C}(G))$ .*

- (i) *Suppose that  $\mathcal{H}_*^?$  has the tree property (4.1). Then the class  $\mathcal{D}_{\text{fib}}(\mathcal{C})$  has the property (TREE) from Definition 0.4.*
- (ii) *Suppose  $\mathcal{H}_*^?$  has the tree property (4.1) and  $\mathcal{C} \subseteq \mathcal{FIN}$ . Then the class  $\mathcal{D}(\mathcal{C})$  has the property (TREE) from Definition 0.4.*
- (iii) *Suppose that  $\mathcal{H}_*^?$  has the regular tree property (4.1) and  $\mathcal{C} \subseteq \mathcal{VCYC}$ . Then the class  $\mathcal{D}(\mathcal{C})$  has the property (TREE $_{\mathcal{R}}$ ) from Definition 0.10.*

*Proof.* Let  $G$  act on a tree  $T$ . Denote by  $V$  the set of vertices of  $T$  and by  $E$  the set of edges. For  $x \in V \cup E$  denote by  $G_x$  the isotropy group of  $x$  and by  $\phi_x: G_x \rightarrow G$  the inclusion. Let  $\mathcal{I}_T = \{H \leq G \mid T^H \neq \emptyset\}$ . Since in a tree there is a unique geodesic between any two points, the fixed set  $T^H$  is contractible for  $H \in \mathcal{I}_T$ . Thus  $T$  is a model for  $E_{\mathcal{I}_T}(G)$ .

Next we prove (i). In this case we assume that for each  $x \in V \cup E$  the pair  $(G_x, \mathcal{C}(G_x))$  satisfies the Fibered Isomorphism Conjecture 1.1. Let  $\phi: K \rightarrow G$  be a group homomorphism. Then  $K$  acts via  $\phi$  on  $T$ . Equipped with this action  $T$  is also a model for  $E_{\phi^*\mathcal{I}_T}(K)$ . The tree property implies that  $(K, \phi^*\mathcal{I}_T)$  satisfies the Isomorphism Conjecture 1.1. Thus  $(G, \mathcal{I}_T)$  satisfies the Fibered Isomorphism Conjecture 1.1. Since the isotropy groups of  $T$  satisfy the Fibered Isomorphism Conjecture 1.1 with respect to  $\mathcal{C}$ , we can deduce from Theorem 2.4 that  $(G, \mathcal{C}(G) \cap \mathcal{I}_T)$  satisfies the Fibered Isomorphism Conjecture 1.1. Finally we use the fact that for the *Fibered* Isomorphism Conjecture 1.1 we can always enlarge the family (see [4, Lemma 1.6]) to conclude that the pair  $(G, \mathcal{C}(G))$  satisfies the Fibered Isomorphism Conjecture 1.1.

Next we prove (ii). In this case we assume that for each  $x \in V \cup E$  the pair  $(G_x, \mathcal{C}(G_x))$  satisfies the Isomorphism Conjecture 1.1. Arguing as above

we conclude that  $(G, \mathcal{I}_T \cap \mathcal{C}(G))$  satisfies the Isomorphism Conjecture 1.1. Finite groups cannot act without fixed points on trees [26, Theorem 15 in 6.1 on page 58 and 6.3.1 on page 60]. Therefore  $\mathcal{I}_T \cap \mathcal{C}(G) = \mathcal{C}(G)$ .

Finally, we prove (iii). In this case we assume that for each  $x \in V \cup E$  the pair  $(G_x, \mathcal{C}(G_x))$  satisfies the Isomorphism Conjecture 1.1 and that  $G_e$  is regular coherent for each  $e \in E$ . Arguing as before we conclude that  $(G, \mathcal{I}_T \cap \mathcal{C}(G))$  satisfies the Isomorphism Conjecture 1.1. We have to show that  $(G, \mathcal{C}(G))$  satisfies the Isomorphism Conjecture 1.1. Because of Theorem 2.4 it suffices to show for any virtually cyclic group  $V \in \mathcal{C}(G)$  that the Isomorphism Conjecture 1.1 holds for  $(V, \mathcal{I}_T \cap \mathcal{C}(G)|_V) = (V, \mathcal{I}_T)$ .

We first consider the case, where  $V$  contains a non-trivial normal finite subgroup  $F$ . We saw above that  $T^F$  is not empty and contractible. By Lemma 4.3 regular coherent groups are torsionfree. Thus isotropy groups of edges are torsionfree, therefore  $T^F$  is just a single vertex of  $T$ . Since  $F$  is normal in  $V$ , the action of  $V$  leaves the fixed points of  $F$  invariant. Therefore the vertex  $T^F$  is a fixed point for  $V$ . Hence we have  $V \in \mathcal{I}_T$  so that  $\mathcal{I}_T$  consists of all subgroups of  $V$ .

If  $V$  does not contain a non-trivial normal finite subgroup  $F$ , it is either  $\mathbb{Z}$  or the infinite dihedral group. In both cases  $V$  acts on the tree  $\mathbb{R}$  with finite stabilizers such that the stabilizers of the edges are trivial and every finite subgroup of  $V$  occurs as stabilizer. The tree  $\mathbb{R}$  is a model for  $E_{\mathcal{FIN}}(V)$ . Since  $\mathcal{H}_*^?$  has the regular tree property (4.1) the map  $\mathcal{H}_n^V(\mathbb{R}) \rightarrow \mathcal{H}_n^V(\text{pt})$  is bijective for all  $n \in \mathbb{Z}$ . This shows that  $V$  satisfies the Isomorphism Conjecture 1.1 for  $(V, \mathcal{FIN}(V))$ . If  $V = \mathbb{Z}$ , then every subgroup  $H \subseteq V$  is trivial or isomorphic to  $\mathbb{Z}$ . If  $V$  is the infinite dihedral group, then any subgroup  $H$  of  $V$  is finite, infinite cyclic or infinite dihedral. We conclude from Theorem 2.4 that  $V$  satisfies the Isomorphism Conjecture 1.1 for every family which contains  $\mathcal{FIN}$ , in particular for  $\mathcal{I}_T$ .  $\square$

**Lemma 4.3.** *Regular coherent groups are torsionfree.*

*Proof.* Assume that  $F$  is a finite subgroup of a regular coherent group  $G$ . Then the  $\mathbb{Z}G$ -module  $\mathbb{Z}[G/F]$  is finitely presented and has a finite-dimensional resolution by finitely generated projective  $\mathbb{Z}G$ -modules since  $G$  is regular coherent and the ring  $\mathbb{Z}$  is regular. Thus the restriction of  $\mathbb{Z}[G/F]$  to a  $\mathbb{Z}F$ -module has a finite-dimensional resolution by projective (but no longer finite generated)  $\mathbb{Z}F$ -modules. As an  $\mathbb{Z}F$ -module  $\mathbb{Z}[G/F]$  contains  $\mathbb{Z}$  (with the constant  $F$ -action) as a direct summand. Therefore  $\mathbb{Z}$  has a finite-dimensional resolution by projective  $\mathbb{Z}F$ -modules. This is only possible if  $F$  is the trivial group.  $\square$

**Corollary 4.4.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory which has the tree property (see Definition 4.1). Let  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  be an extension of groups. Suppose that  $K$  acts on a tree with finite stabilizers and that  $(Q, \mathcal{FIN})$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).*

Then  $(G, \mathcal{FIN})$  satisfies the Fibered Isomorphism Conjecture 1.1 (in the range  $\leq N$ ).

*Proof.* We first treat the case  $Q = \{1\}$ . Then the claim follows from Theorem 4.2 (i) because for a finite group  $F$  the pair  $(F, \mathcal{FIN})$  obviously satisfies the Fibered Isomorphism Conjecture 1.1.

Next we treat the case, where  $Q$  is finite. By a result of Dunwoody [11, Theorem 1.1] a group  $K$  acts on a tree with finite stabilizers if and only if  $H^p(K; \mathbb{Q}) = 0$  for each  $p \geq 2$ . Since  $K$  acts on a tree with finite stabilizers, the trivial  $\mathbb{Q}K$ -module  $\mathbb{Q}$  has a 1-dimensional projective resolution. Hence the trivial  $\mathbb{Q}G$ -module  $\mathbb{Q}$  has a 1-dimensional projective resolution since  $[G : K]$  is finite and invertible in  $\mathbb{Q}$ . This implies  $H^p(G; \mathbb{Q}) = 0$  for each  $p \geq 2$ . Hence also  $G$  acts on a tree with finite stabilizers if  $Q$  is finite. This proves the claim for finite  $Q$ .

Now the general case follows from Lemma 2.5.  $\square$

**Lemma 4.5.** *Let  $\mathcal{H}_*^?$  be an equivariant homology theory which is continuous. Then the following assertions are equivalent.*

- (i) *For each 1-dimensional  $G$ -CW-complex  $T$  for which each component is contractible (after forgetting the group action), the projection  $\text{pr}_T: T \rightarrow \pi_0(T)$  induces isomorphisms*

$$\mathcal{H}_n^G(\text{pr}_T): \mathcal{H}_n^G(T) \xrightarrow{\cong} \mathcal{H}_n^G(\pi_0(T)),$$

*for each  $n \in \mathbb{Z}$ , where we consider  $\pi_0(T)$  as a  $G$ -space using the discrete topology;*

- (ii)  *$\mathcal{H}_*^?$  has the tree property, i.e. for each 1-dimensional  $G$ -CW-complex  $T$ , which is contractible (after forgetting the group action), and each  $n \in \mathbb{Z}$  we obtain isomorphisms*

$$\mathcal{H}_n^G(\text{pr}_T): \mathcal{H}_n^G(T) \xrightarrow{\cong} \mathcal{H}_n^G(\text{pt});$$

- (iii) *For each 1-dimensional  $G$ -CW-complex  $X$ , which is contractible (after forgetting the group action) and has only one equivariant 1-cell, and each  $n \in \mathbb{Z}$  we obtain isomorphisms*

$$\mathcal{H}_n^G(\text{pr}_X): \mathcal{H}_n^G(X) \xrightarrow{\cong} \mathcal{H}_n^G(\text{pt}).$$

*These three assertions remain equivalent if we add the requirement that the isotropy groups of edges are regular coherent to each assertion. (Thus (ii) becomes the assertion that  $\mathcal{H}_*^?$  has the regular tree property.)*

*Proof.* (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) is obvious.

(iii)  $\Rightarrow$  (i) We prove the claim first under the assumption that  $G \backslash T$  has finitely many 1-cells.

We use induction over the number of 1-cells in  $G \backslash T$ . In the induction beginning, where  $G \backslash T$  has no 1-cell,  $T$  is the disjoint union of homogeneous spaces and the claim follows from the fact that  $\mathcal{H}_*^?$  satisfies the disjoint union axiom.

In the induction step we can write  $T$  as a  $G$ -pushout

$$\begin{array}{ccc} G/H \times S^0 & \xrightarrow{q} & T_0 \\ \downarrow & & \downarrow \\ G/H \times D^1 & \longrightarrow & T \end{array}$$

for a  $G$ -CW-subcomplex  $T_0 \subseteq T$  such that  $G \backslash T_0$  has one 1-cell less than  $G \backslash T$ . Here  $H$  is the isotropy group of the 1-cell of  $T$  that is not contained in  $T_0$ . Since a connected subgraph of a tree is again a tree, each component of  $T_0$  is contractible. The induction hypothesis applies to  $T_0$ ,  $G/H \times S^0$  and  $G/H \times D^1$ . Define  $X$  to be the  $G$ -pushout

$$\begin{array}{ccc} G/H \times S^0 & \xrightarrow{\text{pr}_{T_0} \circ q} & \pi_0(T_0) \\ \downarrow & & \downarrow \\ G/H \times D^1 & \longrightarrow & X \end{array}$$

The  $G$ -maps  $\text{pr}_{T_0}: T_0 \rightarrow \pi_0(T_0)$ ,  $\text{id}_{G/H \times S^0}$  and  $\text{id}_{G/H \times D^1}$  are non-equivariant homotopy equivalences and induce a  $G$ -map  $f: T \rightarrow X$  which is a non-equivariant homotopy equivalence since  $G/H \times S^0 \rightarrow G/H \times D^1$  is a cofibration. In particular  $X$  is a 1-dimensional  $G$ -CW-complex whose components are contractible. By a Mayer-Vietoris argument and the Five-Lemma the map

$$\mathcal{H}_n^G(f): \mathcal{H}_n^G(T) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective for all  $n \in \mathbb{Z}$ . The following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{\text{pr}_T} & \pi_0(T) \\ f \downarrow & & \downarrow \pi_0(f) \\ X & \xrightarrow[\text{pr}_X]{} & \pi_0(X) \end{array}$$

Since the map  $\pi_0(f)$  is bijective and hence a  $G$ -homeomorphism,  $\mathcal{H}_n^G(\pi_0(f))$  is bijective for all  $n \in \mathbb{Z}$ . Recall that we have to show that  $\mathcal{H}_n^G(\text{pr}_T)$  is bijective for all  $n \in \mathbb{Z}$ . Hence it suffices to show that  $\mathcal{H}_n^G(\text{pr}_X)$  is bijective for all  $n \in \mathbb{Z}$ . This follows from the fact that we can write  $X$  as a disjoint union of a  $G$ -CW-complex  $Y$ , for which the assumption (iii) applies, and a 0-dimensional  $G$ -CW-complex  $Z$ , for which the induction beginning applies, and that  $\mathcal{H}_*^?$  satisfies the disjoint union axiom.



Next we treat the general case. Because  $\mathcal{H}_*^?$  satisfies the disjoint union axiom, we can assume without loss of generality that  $G \backslash T$  is connected. Since we can write  $T = G \times_H T'$  for a path component  $T'$  and we have natural isomorphisms  $\mathcal{H}^H(T') \xrightarrow{\cong} \mathcal{H}_n^G(T)$  and  $\mathcal{H}^H(\text{pt}) \xrightarrow{\cong} \mathcal{H}_n^G(G/H)$ , we can assume without loss of generality that  $T$  is contractible.

Fix a 0-cell  $e \in G \backslash T$ . Let  $I$  be the set of finite connected  $CW$ -subcomplexes  $Z \subseteq G \backslash T$  with  $e \in Z$ . It can be directed by inclusion and satisfies  $G \backslash T = \bigcup_{Z \in I} Z$ . Let  $p: T \rightarrow G \backslash T$  be the projection. Then  $T$  is the directed union of the  $G$ - $CW$ -subcomplexes  $p^{-1}(Z)$ . Because of Lemma 2.1 the canonical map

$$\text{colim}_{Z \in I} \mathcal{H}_n^G(p^{-1}(Z)) \xrightarrow{\cong} \mathcal{H}_n^G(T)$$

is bijective. Since each  $G$ - $CW$ -complex  $p^{-1}(Z)$  has only finitely many equivariant 1-cells and hence satisfies the claim, and a colimit of a system of isomorphisms is again an isomorphism, it suffices to show that

$$\text{colim}_{Z \in I} \mathcal{H}_n^G(\pi_0(p^{-1}(Z))) \rightarrow \mathcal{H}_n^G(\text{pt}) \quad (4.6)$$

is bijective. Fix  $\tilde{e} \in T$  with  $p(\tilde{e}) = e$ . Let  $G_Z$  be the isotropy group of the path component of  $p^{-1}(Z)$  containing  $\tilde{e}$  in the  $G$ -set  $\pi_0(p^{-1}(Z))$ . Since each  $Z$  is connected,  $\pi_0(p^{-1}(Z))$  is  $G/G_Z$ . We have for any inclusion  $Z_1 \subseteq Z_2$  for elements  $Z_1, Z_2 \in I$ , that  $G_{Z_1}$  is a subgroup of  $G_{Z_2}$ . We have  $G = \bigcup_{Z \in I} G_Z$ . Since  $\mathcal{H}_*^?$  is continuous, we get an isomorphism

$$\text{colim}_{Z \in I} \mathcal{H}_n^G(G/G_Z) \xrightarrow{\cong} \mathcal{H}_n^G(\text{pt}).$$

But this isomorphism can easily be identified with the map (4.6). This finishes the proof of Lemma 4.5.  $\square$

**Remark 4.7.** Let  $G$  act on a tree  $T$ , such that  $G \backslash T$  has only finitely many 1-cells. The proof of Lemma 4.5 shows that then  $G$  acts on tree  $X$  with the following properties: The quotient  $G \backslash X$  has only one 1-cell. For each edge  $e$  of  $X$  the isotropy group  $G_e$  is also the isotropy group of an edge  $e'$  of  $T$ . For each vertex  $v$  of  $X$  there is a subtree  $T_v$  of  $T$  that is invariant under the isotropy group  $G_v$  and for which  $G_v \backslash T_v$  has one less 1-cell than  $G \backslash T$ . In combination with the colimit argument from the proof of Lemma 4.5 this means that a class of groups  $\mathcal{C}$  that has property (COL) from Definition 0.4 has property (TREE) from Definition 0.4 if and only if it has the following property

**(TREE')** Suppose that  $G$  acts on a tree  $T$  where  $T$  has only one equivariant 1-cell. Assume that for each  $x \in T$  the isotropy group  $G_x$  belongs to  $\mathcal{C}$ . Then  $G$  belongs to  $\mathcal{C}$ ;

and has property (TREE $_{\mathcal{R}}$ ) if and only if it has the property

**(TREE' <sub>$\mathcal{R}$</sub> )** Suppose that  $G$  acts on a tree  $T$ . Assume that for each  $x \in T$  the isotropy group  $G_x$  belongs to  $\mathcal{C}$ . For each edge  $e$  of  $T$ , assume that the isotropy group  $G_e$  is regular coherent. Then  $G$  belongs to  $\mathcal{C}$ .

Note on the other hand, that the statement that the Fibered Isomorphism Conjecture 1.1 has property (TREE') is really a statement about arbitrary actions on trees: If  $G$  acts on a tree  $T$  where  $T$  has only one equivariant 1-cell and  $\phi: K \rightarrow G$  is a group homomorphism, then the induced action of  $K$  on  $T$  may have more equivariant 1-cells and may even be no longer cocompact. Therefore we have to consider general trees in the formulation of the tree property in Definition 4.1.

## 5 Equivariant homology theories constructed from spectra

In this section we want to give a criterion when an equivariant homology theory has the tree property provided that it arises from a covariant functor  $\mathbf{E}: \mathbf{GROUPOIDS} \rightarrow \mathbf{SPECTRA}$  which sends equivalences of groupoids to weak equivalences of spectra. This will be the main example for us.

Fix a group  $G$ . The *transport groupoid*  $\mathcal{G}^G(S)$  of a  $G$ -set  $S$  has  $S$  as set of objects and the set of morphism from  $s_1$  to  $s_2$  consists of those element  $g \in G$  with  $gs_1 = s_2$ . Composition of morphisms comes from the group structure on  $G$ . The orbit category  $\mathbf{Or}(G)$  has as objects homogeneous spaces  $G/H$  and as morphisms  $G$ -maps. We obtain a covariant functor  $\mathcal{G}^G: \mathbf{Or}(G) \rightarrow \mathbf{GROUPOIDS}$ ,  $G/H \mapsto \mathcal{G}^G(G/H)$ . Define the covariant functor  $\mathbf{E}^G: \mathbf{Or}(G) \rightarrow \mathbf{SPECTRA}$  by  $\mathbf{E} \circ \mathcal{G}^G$ . Let  $H_*^G(-; \mathbf{E})$  be the  $G$ -homology theory associated to  $\mathbf{E}^G$  in [10, Section 4 and Section 7]. It is not hard to construct the relevant induction structure to get an equivariant homology theory  $H_*^?(-; \mathbf{E})$ . It has the property that for each group  $G$  with subgroup  $H \subseteq G$  and each  $n \in \mathbb{Z}$  we have canonical isomorphisms

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\mathrm{pt}; \mathbf{E}) \cong \pi_n(\mathbf{E}(H)).$$

In the expression  $\mathbf{E}(H)$  we think of the group  $H$  as a groupoid with one object. More details of the construction of  $H_*^?(-; \mathbf{E})$  can be found in [18] and [25].

**Lemma 5.1.** *The equivariant homology theory  $H_*^?(-, \mathbf{E})$  is continuous and has the tree property if and only if the following conditions are satisfied*

- (i) *For each group  $G$  and directed system of subgroups  $\{G_i \mid i \in I\}$ , which is directed by inclusion and satisfies  $\bigcup_{i \in I} G_i = G$ , and each  $n \in \mathbb{Z}$  the map*

$$\mathrm{colim}_{i \in I} j_i: \mathrm{colim}_{i \in I} \pi_n(\mathbf{E}(G_i)) \rightarrow \pi_n(\mathbf{E}(G))$$

*is an isomorphism, where  $j_i$  is the homomorphism induced by the inclusion  $G_i \rightarrow G$ ;*

(ii) Consider a pushout of groups

$$\begin{array}{ccc}
H_0 & \xrightarrow{i_1} & H_1 \\
i_2 \downarrow & & \downarrow j_1 \\
H_2 & \xrightarrow{j_2} & G
\end{array} \tag{5.2}$$

such that  $i_1$  and  $i_2$  are injective. In other words,  $G$  is the amalgamated product of  $H_1$  and  $H_2$  over  $H_0$  with respect to the injections  $i_1$  and  $i_2$ . Then for each such pushout (5.2) the following square of spectra is homotopy cocartesian

$$\begin{array}{ccc}
\mathbf{E}^G(G/H_0) \vee \mathbf{E}^G(G/H_2) & \xrightarrow{\mathbf{E}^G(\text{pr}_1) \vee \mathbf{E}^G(\text{pr}_2)} & \mathbf{E}^G(G/H_1) \vee \mathbf{E}^G(G/H_2) \\
\text{id} \vee \text{id} \downarrow & & \mathbf{E}^G(\text{pr}_3) \vee \mathbf{E}^G(\text{pr}_4) \downarrow \\
\mathbf{E}^G(G/H_0) & \xrightarrow{\mathbf{E}^G(\text{pr}_5)} & \mathbf{E}^G(G/G)
\end{array} \tag{5.3}$$

where the maps labeled  $\text{pr}_i$  denote canonical projections.

(iii) Let  $i_0, i_1: H \rightarrow K$  be injective group homomorphisms. Let  $G$  be the HNN-extension associated to  $i_0$  and  $i_1$ . The HNN-extension comes with an inclusion  $j: K \rightarrow G$  and  $t \in G$  such that  $j \circ i_0 = c_t \circ j \circ i_1$ , where  $c_t$  is conjugation by  $t$ . (This is the defining property of the HNN-extension.) We will use  $i_0$  to consider  $H$  as a subgroup of  $G$ . Then the following square of spectra is homotopy cocartesian

$$\begin{array}{ccc}
\mathbf{E}^G(G/H) \vee \mathbf{E}^G(G/H) & \xrightarrow{\mathbf{E}^G(\text{pr}_0) \vee \mathbf{E}^G(\beta)} & \mathbf{E}^G(G/K) \\
\text{id} \vee \text{id} \downarrow & & \downarrow \mathbf{E}^G(\text{pr}_1) \\
\mathbf{E}^G(G/H) & \xrightarrow{\mathbf{E}^G(\text{pr}_2)} & \mathbf{E}^G(G/G)
\end{array} \tag{5.4}$$

where the maps labeled  $\text{pr}_i$  are canonical projections while  $\beta$  is defined by  $\beta(gH) = gtK$ .

For conditions (ii) and (iii) one can also consider the regular versions where (5.3) is only required to be homotopy cartesian if in addition  $H_0$  is regular coherent and torsionfree and (5.4) is only required to be homotopy cartesian if in addition  $H$  is regular coherent and torsionfree. Then the equivariant homology theory  $H_*^?(-, \mathbf{E})$  is continuous and has the regular tree property if and only the condition (i) and the regular versions of conditions (ii) and (iii) hold.

*Proof.* Obviously condition (i) is equivalent to the condition that  $\mathcal{H}_*^?$  is continuous. From now on we assume that  $\mathcal{H}_*^?$  is continuous.

Suppose that the two conditions (ii) and (iii) are satisfied. Because of Lemma 4.5 it suffices to prove the tree property only for 1-dimensional contractible  $G$ -CW-complexes  $T$  such that there is precisely one equivariant 1-cell. Such a  $G$ -CW-complex will have precisely one or precisely two equivariant 0-cells. We only treat the case, where there are two equivariant 0-cells, the proof of the other case is analogous using condition (iii) instead of condition (ii).

We can write  $T$  as a  $G$ -pushout

$$\begin{array}{ccc} G/H_0 \times S^0 & \xrightarrow{\text{pr}_1 \amalg \text{pr}_2} & G/H_1 \amalg G/H_2 \\ \downarrow & & \downarrow \\ G/H_0 \times D^1 & \longrightarrow & T \end{array}$$

where  $H_0$  is a subgroup of both  $H_1$  and  $H_2$  and  $\text{pr}_1$  and  $\text{pr}_2$  are the canonical projections. Recall that a  $G$ -space  $Z$  defines a contravariant functor  $\text{Or}(G) \rightarrow \text{SPACES}$ ,  $G/H \mapsto \text{map}_G(G/H, Z)$  and that we get a spectrum  $\text{map}_G(G/?, Z) \wedge_{\text{Or}(G)} \mathbf{E}^G$  by the tensor product over the orbit category (see [10, Section 1]). If we apply  $\text{map}_G(G/?, -) \wedge_{\text{Or}(G)} \mathbf{E}^G$  to the  $G$ -pushout above, we obtain a homotopy cocartesian diagram of spectra

$$\begin{array}{ccc} \mathbf{E}^G(G/H_0) \vee \mathbf{E}^G(G/H_0) & \xrightarrow{\mathbf{E}^G(\text{pr}_1) \vee \mathbf{E}^G(\text{pr}_2)} & \mathbf{E}^G(G/H_1) \vee \mathbf{E}^G(G/H_2) \\ \text{id} \vee \text{id} \downarrow & & \downarrow \\ \mathbf{E}^G(G/H_0) & \longrightarrow & \text{map}_G(G/?, T) \wedge_{\text{Or}(G)} \mathbf{E}^G \end{array} \quad (5.5)$$

The following diagram is a pushout of groups

$$\begin{array}{ccc} H_0 & \xrightarrow{i_1} & H_1 \\ i_2 \downarrow & & \downarrow j_1 \\ H_2 & \xrightarrow{j_2} & G \end{array}$$

where  $i_k: H_0 \rightarrow H_k$ ,  $j_k: H_k \rightarrow G$  are inclusion (see [26, Example 1 on page 43]). Hence by condition (ii) we have the homotopy cocartesian square (5.3). The projection  $\text{pr}: T \rightarrow G/G$  induces a map from the right lower corner of the diagram (5.5) to the right lower corner of the diagram (5.3), if we identify  $G/G \wedge_{\text{Or}(G)} \mathbf{E}^G = \mathbf{E}^G(G/G)$ . If we take the identity on the other three corners, we get a map between homotopy cocartesian squares of spectra. Since the three identity maps are obviously weak equivalences, the fourth map induced by the projection is a weak equivalence. But this map induces on homotopy groups the map  $H_n^G(\text{pr}; \mathbf{E}): H_n^G(T; \mathbf{E}) \rightarrow H_n^G(\text{pt})$  which is hence bijective for each  $n \in \mathbb{Z}$ .

This shows that  $H_*^?(-; \mathbf{E})$  has the tree property if conditions (ii) and (iii) are satisfied. It is now also obvious that conditions (ii) and (iii) hold if  $H_*^?(-; \mathbf{E})$  has the tree property.  $\square$

**Remark 5.6.**

- (i) In the situation of Lemma 5.1 (ii) diagram (5.3) is homotopy cocartesian if and only if the commutative diagram

$$\begin{array}{ccc} \mathbf{E}(H) \vee \mathbf{E}(H) & \xrightarrow{\mathbf{E}(i_1) \vee \mathbf{E}(i_2)} & \mathbf{E}(G_1) \vee \mathbf{E}(G_2) \\ \text{id} \vee \text{id} \downarrow & & \downarrow \mathbf{E}(j_1) \vee \mathbf{E}(j_2) \\ \mathbf{E}(H) & \xrightarrow{\mathbf{E}(j_0)} & \mathbf{E}(G) \end{array}$$

where  $j_0: H \rightarrow G$  is defined to be  $j_1 \circ i_1 = j_2 \circ i_2$ , is homotopy cocartesian since there is a canonical weak equivalence from each corner of this square to the corresponding corner of (5.3).

- (ii) The situation in Lemma 5.1 (iii) is a bit more complicated. The natural diagram to consider is

$$\begin{array}{ccc} \mathbf{E}(H) \vee \mathbf{E}(H) & \xrightarrow{\mathbf{E}(i_0) \vee \mathbf{E}(i_1)} & \mathbf{E}(K) \\ \text{id} \vee \text{id} \downarrow & & \downarrow \mathbf{E}(j) \\ \mathbf{E}(H) & \xrightarrow{\mathbf{E}(j \circ i_0)} & \mathbf{E}(G) \end{array} \quad (5.7)$$

However, (5.7) is not commutative, while (5.4) is commutative. There is a canonical weak equivalence from each corner of (5.4) to the corresponding corner of (5.7), but those maps do not make the square

$$\begin{array}{ccc} \mathbf{E}(H) \vee \mathbf{E}(H) & \xrightarrow{\mathbf{E}(i_0) \vee \mathbf{E}(i_1)} & \mathbf{E}(K) \\ \downarrow & & \downarrow \\ \mathbf{E}^G(G/H) \vee \mathbf{E}^G(G/H) & \xrightarrow{\mathbf{E}^G(\text{pr}_0) \vee \mathbf{E}^G(\beta)} & \mathbf{E}^G(G/K) \end{array}$$

commutative.

The failure of the commutativity of (5.7) stems from the fact, that the underlying diagram of groups commutes only up to conjugation, i.e.  $j \circ i_0 \neq j \circ i_1 = c_t \circ j \circ i_0$ . It is a consequence of the definitions that  $\mathbf{E}(c_t)$  is weakly homotopic to  $\text{id}_{\mathbf{E}(G)}$ , but in general there is no preferred homotopy. On the other hand  $\mathbf{E}: \mathbf{GROUPOIDS} \rightarrow \mathbf{SPECTRA}$  is often slightly better than required in the discussion before Lemma 5.1, namely  $\mathbf{E}$  is a 2-functor. This means that if  $\tau$  is a natural transformation between functors  $f, g$  between groupoids, then there is a (preferred) homotopy  $\mathbf{E}(\tau)$  from  $\mathbf{E}(f)$  to  $\mathbf{E}(g)$ . Under this stronger assumption on  $\mathbf{E}$  there is a canonical homotopy that makes (5.7) homotopy commutative and then condition (iii) in Lemma 5.1 is equivalent to requiring that (5.7) is homotopy cocartesian with respect to the canonical homotopy.

## 6 Isomorphism Conjectures for spectra

In this section we relate the (Fibered) Isomorphism Conjecture 1.1 for an equivariant homology theory  $\mathcal{H}_*^?$  to the versions appearing in Farrell-Jones [12] for algebraic  $K$ - and  $L$ -theory.

Consider a group homomorphism  $\phi: K \rightarrow G$ , a  $K$ -CW-complex  $Z$  and a covariant functor  $\mathbf{E}: \text{SPACES} \rightarrow \text{SPECTRA}$ , which sends weak equivalences to weak equivalences and is *compatible with disjoint unions*, i.e. for a family  $\{Y_i \mid i \in I\}$  of spaces the map induced by the inclusions  $j_i: Y_i \rightarrow \coprod_{i \in I} Y_i$

$$\bigvee_{i \in I} \mathbf{E}(j_i): \bigvee_{i \in I} \mathbf{E}(Y_i) \rightarrow \mathbf{E}\left(\coprod_{i \in I} Y_i\right)$$

is a weak equivalence. We obtain a covariant functor

$$\mathbf{E}_Z^K: \text{Or}(K) \rightarrow \text{SPECTRA}, \quad K/H \mapsto \mathbf{E}(Z \times_K K/H).$$

Recall that for each covariant functor  $\mathbf{F}: \text{Or}(K) \rightarrow \text{SPECTRA}$  there is a  $K$ -homology theory  $H_*^K(-; \mathbf{F})$  defined for  $K$ -CW-complexes with the property that  $H_n^K(K/H; \mathbf{F}) \cong \pi_n(\mathbf{F}(K/H))$  holds for  $H \subseteq K$  and  $n \in \mathbb{Z}$ . [10, Section 4 and Section 7]. We denote by  $\phi_*Z$  the  $G$ -space  $G \times_\phi Z$  obtained by induction with  $\phi$  from the  $K$ -space  $Z$ . For a  $G$ -space  $X$  let  $\phi^*X$  be the  $K$ -space obtained by restricting the  $G$ -action to a  $K$ -action using  $\phi$ .

**Lemma 6.1.** *For any  $G$ -CW-complex  $X$  there is an isomorphism, natural in  $X$ ,  $Z$  and  $\mathbf{E}$ ,*

$$\phi_*: H_n^K(\phi^*X; \mathbf{E}_Z^K) \xrightarrow{\cong} H_n^G(X; \mathbf{E}_{\phi_*Z}^G).$$

*Proof.* Let  $\phi: \text{Or}(K) \rightarrow \text{Or}(G)$ ,  $K/H \mapsto G/\phi(H)$  be the functor induced by  $\phi$ . Given a contravariant (pointed)  $\text{Or}(G)$ -space  $A$  and a covariant (pointed)  $\text{Or}(K)$ -space  $B$  there is an adjunction

$$\text{res}_\phi A \otimes_{\text{Or}(K)} B \xrightarrow{\cong} A \otimes_{\text{Or}(G)} \text{ind}_\phi B, \quad (6.2)$$

where  $\text{res}_\phi$  is restriction and  $\text{ind}_\phi$  denotes induction with the functor  $\phi: \text{Or}(K) \rightarrow \text{Or}(G)$  (see [10, Lemma 1.9]). It induces a natural isomorphism

$$H_n^K(\phi^*X; \mathbf{E}_Z^K) \xrightarrow{\cong} H_n^G(X; \text{ind}_\phi \mathbf{E}_Z^K)$$

There is a weak equivalence of covariant  $\text{Or}(G)$ -spectra

$$\text{ind}_\phi \mathbf{E}_Z^K \xrightarrow{\cong} \mathbf{E}_{\phi_*Z}^G$$

coming from

$$\begin{aligned} \text{map}_G(G/\phi(?), G/??) \otimes_{\text{Or}(K)} (Z \times_K K/?) &\xrightarrow{\cong} Z \times_\phi G/??, \\ (f, (z, k?)) &\mapsto (z, f(\phi(k)\phi(?))) \end{aligned}$$

and the fact that  $\mathbf{E}$  is compatible with disjoint unions.  $\square$

**Lemma 6.3.** *Let  $\mathcal{F}$  be a family of subgroups of  $G$ . Let  $N \in \mathbb{Z}$ . Then the following assertions are equivalent:*

- (i) *For any free  $G$ -CW-complex  $Z$  and  $n \in \mathbb{Z}$  (with  $n \leq N$ ) the assembly map*

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{E}_Z^G) \xrightarrow{\cong} H_n^G(\text{pt}; \mathbf{E}_Z^G)$$

*is bijective;*

- (ii) *For each injective group homomorphism  $\phi: K \rightarrow G$  and any free connected  $K$ -CW-complex  $Z$  and  $n \in \mathbb{Z}$  (with  $n \leq N$ ) the assembly map*

$$H_n^K(E_{\phi^*(\mathcal{F})}(K); \mathbf{E}_Z^K) \xrightarrow{\cong} H_n^K(\text{pt}; \mathbf{E}_Z^K)$$

*is bijective;*

- (iii) *For each group homomorphism  $\phi: K \rightarrow G$  and any free simply connected  $K$ -CW-complex  $Z$  and  $n \in \mathbb{Z}$  (with  $n \leq N$ ) the assembly map*

$$H_n^K(E_{\phi^*(\mathcal{F})}(K); \mathbf{E}_Z^K) \xrightarrow{\cong} H_n^K(\text{pt}; \mathbf{E}_Z^K)$$

*is bijective.*

*Proof.* (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii) These implications follow from Lemma 6.1 since for any group homomorphism  $\phi: K \rightarrow G$  we have  $\phi^*(E_{\mathcal{F}}(G)) = E_{\phi^*(\mathcal{F})}(K)$ .

(ii)  $\Rightarrow$  (i) We can write a  $G$ -CW-complex  $Z$  as  $Z = \coprod_{i \in I} G \times_{G_i} Z_i$  for subgroups  $G_i \subseteq G$  and connected free  $G_i$ -CW-complexes  $Z_i$ . Since  $\mathbf{E}$  is compatible with disjoint unions, we conclude from [10, Lemma 4.6] that we can assume without loss of generality that  $I$  consists of one element 0, i.e.  $Z = G \times_{G_0} Z_0$ . Now the claim follows from Lemma 6.1 applied to the inclusion  $\phi: G_0 \rightarrow G$  and the free connected  $G_0$ -CW-complex  $Z_0$ .

(iii)  $\Rightarrow$  (ii) There is an extension of groups  $1 \rightarrow \pi_1(Z) \rightarrow \tilde{K} \xrightarrow{p} K \rightarrow 1$  and a  $\tilde{K}$ -action on the universal covering  $\tilde{Z}$  which extends the  $\pi_1(Z)$ -action on  $\tilde{Z}$  and covers the  $K$ -action on  $Z$ . Moreover,  $\tilde{Z}$  inherits the structure of a free  $\tilde{K}$ -CW-complex. Now the claim follows from Lemma 6.1 applied to the group homomorphism  $p: \tilde{K} \rightarrow K$  and the simply connected free  $\tilde{K}$ -CW-complex  $\tilde{Z}$  since  $p_*\tilde{Z} = Z$  and  $p^*(\phi^*\mathcal{F}) = (\phi \circ p)^*\mathcal{F}$ .  $\square$

**Lemma 6.4.** *Suppose that for any two-connected map  $f: X \rightarrow Y$  the induced map  $\mathbf{E}(f): \mathbf{E}(X) \rightarrow \mathbf{E}(Y)$  is a weak equivalence. Let  $Z$  be a simply-connected  $G$ -CW-complex and  $f: Z \rightarrow EG$  be the classifying map.*

*Then it induces a weak equivalence of  $\text{Or}(G)$ -spectra*

$$\mathbf{f}: \mathbf{E}_Z^G \rightarrow \mathbf{E}_{EG}^G$$

*and in particular for each  $G$ -CW-complex and each  $n \in \mathbb{Z}$  a natural isomorphism*

$$H_n^G(X; \mathbf{E}_Z^G) \xrightarrow{\cong} H_n^G(X; \mathbf{E}_{EG}^G).$$

*Proof.* The map  $f: Z \rightarrow EG$  is 2-connected. Hence the induced map  $f \times_G \text{id}_{G/H}: Z \times_G G/H \rightarrow EG \times_G G/H$  is 2-connected for all subgroups  $H \subseteq G$ . Now apply [10, Lemma 4.6].  $\square$

**Definition 6.5 (Fibred Isomorphism Conjecture for spectra).** We say that a group  $G$  satisfies the *Isomorphism Conjecture for  $\mathcal{F}$  and  $\mathbf{E}$  (in the range  $\leq N$ )* if the assembly map induced by the projection  $\text{pr}: E_{\mathcal{F}}(G) \rightarrow \text{pt}$

$$\text{asmb}: H_n^G(E_{\mathcal{F}}(G); \mathbf{E}_{EG}^G) \rightarrow H_n^G(\text{pt}; \mathbf{E}_{EG}^G)$$

is bijective for all  $n \in \mathbb{Z}$  (with  $n \leq N$ ).

We say that a group  $G$  satisfies the *Fibred Isomorphism Conjecture for  $\mathcal{F}$  and  $\mathbf{E}$  (in the range  $\leq N$ )* if for any free  $G$ -CW-complex  $Z$  the assembly map induced by the projection  $\text{pr}: E_{\mathcal{F}}(G) \rightarrow \text{pt}$

$$\text{asmb}: H_n^G(E_{\mathcal{F}}(G); \mathbf{E}_Z^G) \rightarrow H_n^G(\text{pt}; \mathbf{E}_Z^G)$$

is bijective for all  $n \in \mathbb{Z}$  (with  $n \leq N$ ).

**Remark 6.6.** The (Fibred) Isomorphism Conjecture of Farrell-Jones [12] for algebraic  $K$ -theory or  $L$ -theory respectively is equivalent to the (Fibred) Isomorphisms Conjecture 6.5 for  $(G, \mathcal{VCC}, \mathbf{E})$  if  $\mathbf{E}: \text{SPACES} \rightarrow \text{SPECTRA}$  sends  $X$  to the non-connective algebraic  $K$ -theory spectrum or  $L^{(-\infty)}$ -theory spectrum of the fundamental groupoid of  $X$  respectively. Since a 2-connected map  $f: X \rightarrow Y$  induces an equivalence on the fundamental groupoids, Lemma 6.4 applies. Let  $R$  be a ring. Consider the covariant functors

$$\begin{aligned} \mathbf{K}_R: \text{GROUPOIDS} &\rightarrow \text{SPECTRA} \\ \mathbf{L}_R: \text{GROUPOIDS} &\rightarrow \text{SPECTRA} \end{aligned}$$

defined in [10, Section 2] satisfying  $\pi_n(\mathbf{K}_R(G)) = K_n(RG)$  and  $\pi_n(\mathbf{L}_R(G)) = L_n^{(-\infty)}(RG)$  for each group  $G$  and  $n \in \mathbb{Z}$ . Let  $H_*^?(-, \mathbf{K}_R)$  and  $H_*^?(-, \mathbf{L}_R)$  be the associated equivariant homology theories. Then the (Fibred) Isomorphism Conjecture for algebraic  $K$ -theory or algebraic  $L$ -theory respectively for the group  $G$  in the sense of Farrell-Jones [12] is equivalent to the (Fibred) Isomorphisms Conjecture 1.1 for  $H_*^?(-, \mathbf{K}_R)$  and  $H_*^?(-, \mathbf{L}_R)$  for the pair  $(G, \mathcal{VCC})$ . This follows from Lemma 6.3 and Lemma 6.4.

For more information about the various conjectures such as the version for pseudoisotopy or the Baum-Connes Conjecture we refer for instance to [18].

## 7 The KH-Isomorphism Conjecture

In this section we will formulate the *KH-Isomorphism Conjecture*. The construction of homotopy algebraic  $K$ -Theory is a simplicial construction, so we will quickly fix the notation. The category  $\Delta$  has as objects finite ordered



sets of the form  $\underline{n} = \{0 < 1 < \dots < n\}$  and order preserving maps as morphisms. The  $n$ -simplex  $\Delta_{\bullet}^n$  is the simplicial set  $\underline{m} \mapsto \text{Map}_{\Delta}(\underline{m}, \underline{n})$ . Let  $R$  be a ring. The simplicial ring  $R[\bullet]$  is defined by

$$R[\underline{n}] = R[t_0, \dots, t_n] / (t_0 + \dots + t_n = 1).$$

Here the structure maps acts as follows: if  $f: \underline{n} \rightarrow \underline{m}$  is order preserving then  $f^*: R[\underline{m}] \rightarrow R[\underline{n}]$  is defined by

$$f^*(t_k) = \sum_{j \in f^{-1}(k)} t_j.$$

In [31] the homotopy algebraic  $K$ -theory  $KH_*(R)$  of  $R$  is defined as the homotopy groups of the realization  $\mathbf{K}HR$  of the simplicial spectrum  $\mathbf{K}^{-\infty}R[\bullet]$ . Here  $\mathbf{K}^{-\infty}$  denotes the non-connected  $K$ -theory spectrum; a construction is reviewed before Definition 9.4. To illustrate the construction of homotopy algebraic  $K$ -theory we give a proof of the following fundamental property of homotopy algebraic  $K$ -theory, cf. [31, 1.2.(i)].

**Proposition 7.1.** *The inclusion  $R \hookrightarrow R[X]$  gives an isomorphism  $KH_n(R) \cong KH_n(R[X])$  for all  $n \in \mathbb{Z}$ .*

*Proof.* It suffices to show that  $R[\bullet] \rightarrow R[X][\bullet]$  is a homotopy equivalence of simplicial rings (cf. Remark 7.2). To see this we need to show that  $X \mapsto 0$  is homotopic to the identity of  $R[X][\bullet]$ . Such a homotopy  $R[X][\bullet] \times \Delta_{\bullet}^1 \rightarrow R[X][\bullet]$  is given by

$$(X, f) \mapsto \left( \sum_{j \in f^{-1}(0)} t_j \right) X,$$

where  $f: \underline{n} \mapsto \underline{1}$ . □

**Remark 7.2.** If  $S$  is a ring and  $F$  is a finite set, then  $S \times F$  has a ring structure ( $S \times F \cong \prod_{f \in F} S$ ). Therefore we may view  $R[X][\bullet] \times \Delta_{\bullet}^1$  as a simplicial ring and the above homotopy as a map of simplicial rings. Therefore we get a map  $|\mathbf{K}^{-\infty}(R[X][\bullet] \times \Delta_{\bullet}^1)| \rightarrow |\mathbf{K}^{-\infty}(R[X][\bullet])|$ . On the other hand, there is a map of simplicial spectra  $\mathbf{K}^{-\infty}(R[X][\bullet]) \times \Delta_{\bullet}^1 \rightarrow \mathbf{K}^{-\infty}(R[X][\bullet] \times \Delta_{\bullet}^1)$  defined as follows. For  $f \in \Delta_{\underline{n}}^1$  there is an obvious map of rings  $\iota_f: R[X][\underline{n}] \rightarrow R[X][\underline{n}] \times \{f\} \hookrightarrow R[X][\underline{n}] \times \Delta_{\underline{n}}^1$ . Thus we can map  $(x, f) \in \mathbf{K}^{-\infty}(R[X][\underline{n}] \times \Delta_{\underline{n}}^1)$  to  $\mathbf{K}^{-\infty}(\iota_f)(x) \in \mathbf{K}^{-\infty}(R[X][\underline{n}] \times \Delta_{\underline{n}}^1)$ .

In order to define an equivariant homology theory we define the functor

$$\mathbf{K}H_R: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$$

as the realization of the simplicial functor

$$\mathbf{K}_{R[\bullet]}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}.$$

Thus  $\pi_n(\mathbf{KH}_R(G)) = KH_n(RG)$ . Since the realization of a weak equivalence is again a weak equivalence,  $\mathbf{KH}_R$  sends equivalences of groupoids to weak equivalences of spectra.

**Conjecture 7.3 ((Fibered)  $KH$ -Isomorphism Conjecture).** *A group  $G$  is said to satisfy the (Fibered)  $KH$ -Isomorphism Conjecture (for a ring  $R$ ) if the pair  $(G, \mathcal{FIN})$  satisfies the (Fibered) Isomorphism Conjecture 1.1 for the equivariant homology theory  $H_*^?(-; \mathbf{KH}_R)$ .*

**Remark 7.4.** All virtually cyclic groups act on trees with finite stabilizers. For a finite group  $F$  the (Fibered)  $KH$ -Isomorphism Conjecture holds (since  $E_{\mathcal{FIN}}(F) = F/F$ ). Thus by Theorem 0.5 the (Fibered)  $KH$ -Isomorphism Conjecture holds for virtually cyclic groups. Therefore Theorem 2.4 implies, that it makes no difference if we replace the family of finite groups with the family of virtually cyclic groups in the formulation of the (Fibered)  $KH$ -Isomorphism Conjecture.

## 8 The Relation between the $K$ - and the $KH$ -Isomorphism Conjecture

There is a natural map  $\mathbf{K}^{-\infty}R \rightarrow \mathbf{KH}R$  induced from the inclusion of the constant simplicial ring  $R$  into  $R[\bullet]$ . Similarly we obtain a natural transformation  $\mathbf{K}_R \rightarrow \mathbf{KH}_R$  of functors from  $\mathbf{GROUPOIDS}$  to  $\mathbf{SPECTRA}$ . Thus we obtain a natural transformation of equivariant homology theories  $H_*^?(-; \mathbf{K}_R) \rightarrow H_*^?(-; \mathbf{KH}_R)$  and a commutative diagram between assembly maps

$$\begin{array}{ccc} H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) & \longrightarrow & K_n(RG) \\ \downarrow & & \downarrow \\ H_n^G(E_{\mathcal{FIN}}(G); \mathbf{KH}_R) & \longrightarrow & KH_n(RG) \end{array} \quad (8.1)$$

We first explain what the  $KH$ -Isomorphism Conjecture 7.3 implies for the  $K$ -Isomorphism Conjecture, i.e. the Farrell-Jones Conjecture for algebraic  $K$ -theory (see Remark 6.6). In order to state the connection we need to recall the groups  $N^p K_n(R)$  [6, XIII]. They can be defined by  $N^0 K_n(R) = K_n(R)$  and

$$N^p K_n(R) = \ker(N^{p-1}(q): N^{p-1} K_n(R[t]) \rightarrow N^{p-1} K_n(R)),$$

where  $q(t) = 0$ . For regular rings  $N^p K_n(R) = 0$  for  $p \geq 1$ , see [13].

**Proposition 8.2.** *Let  $G$  be a group that satisfies the  $KH$ -Isomorphism Conjecture 7.3 for the ring  $R$ .*

- (i) *Suppose that  $N^p K_n(RF) = 0$  for all finite subgroups  $F$  of  $G$  and all  $n \in \mathbb{Z}$ ,  $p \geq 1$ . Then the assembly map with respect to the family  $\mathcal{FIN}$  in algebraic  $K$ -theory, i.e. the top row in (8.1), is split injective.*

- (ii) Suppose that  $N^p K_n(RF) \otimes \mathbb{Q} = 0$  for all finite subgroups  $F$  of  $G$  and all  $n \in \mathbb{Z}$ ,  $p \geq 1$ . Then assembly map with respect to the family  $\mathcal{FIN}$  in algebraic  $K$ -theory, i.e. the top row in (8.1), is rationally split injective.

*Proof.* By the spectral sequence from [31, 1.3] the canonical map  $K_*(A) \rightarrow KH_*(A)$  is an isomorphism if  $N^p K_n(A) = 0$  for all  $n \in \mathbb{Z}$  and  $p \geq 1$  and a rational isomorphism if  $N^p K_n(A) \otimes \mathbb{Q} = 0$  for all  $n \in \mathbb{Z}$  and  $p \geq 1$ . Therefore these assumptions imply by a spectral sequence argument that the left vertical map in (8.1) is an isomorphism or a rational isomorphism respectively.  $\square$

**Remark 8.3.** The assumptions of Proposition 8.2 (i) and (ii) are satisfied in the following cases.

- (i) If  $R$  is a regular ring containing  $\mathbb{Q}$  then  $RF$  is regular for all finite groups  $F$ . Thus the assumption in 8.2 (i) is satisfied.
- (ii) If  $R = \mathbb{Z}$  then the assumption in 8.2 (ii) is satisfied. This follows from [30, 6.4], which implies

$$NK_*(\mathbb{Z}[t_1, \dots, t_n]F) \otimes \mathbb{Q} \cong NK_*(\mathbb{Q}[t_1, \dots, t_n]F).$$

Thus for a finite group  $F$  it follows that,  $NK_*(\mathbb{Z}[t_1, \dots, t_n]F)$  vanishes rationally, since  $\mathbb{Q}[t_1, \dots, t_n]F$  is regular. A straight forward induction shows that this implies  $N^p K_n(\mathbb{Z}F) \otimes \mathbb{Q} = 0$  for  $p \geq 1$ .

- (iii) If  $G$  is torsionfree and  $R$  is regular, then the assumption in 8.2 (i) is satisfied. In this case the Farrell-Jones Conjecture in algebraic  $K$ -theory asserts, that the top vertical map in (8.1) is an isomorphism. Thus, in this situation ( $R$  regular,  $G$  torsionfree) the Farrell-Jones Conjecture in algebraic  $K$ -theory holds if  $N_n^p(RG) = 0$  for all  $n \in \mathbb{Z}$  and  $p \geq 1$  and if  $G$  satisfies the  $KH$ -Isomorphism Conjecture.

Next we explain the reverse connection.

**Theorem 8.4 (The  $K$ -theory version implies the  $KH$ -version).** *Let  $G$  be a group and let  $R$  be a ring.*

- (i) *Suppose that the (Fibred) Farrell-Jones Conjecture in algebraic  $K$ -theory is true for  $(G, R[x_1, x_2, \dots, x_n])$  for all  $n \geq 1$  then the (Fibred)  $KH$ -Isomorphism Conjecture 7.3 is true for  $(G, R)$ .*
- (ii) *Suppose that the Fibred Farrell-Jones Conjecture in algebraic  $K$ -theory is true for  $(G \times \mathbb{Z}^n, R)$  for all  $n \geq 1$ . Then the Fibred  $KH$ -Isomorphism Conjecture 7.3 is true for  $(G, R)$ .*

*Proof.* Let  $\phi: K \rightarrow G$  be a group homomorphism. The assembly map  $H_n^K(E_{\phi^* \text{vCyc}}(K); \mathbf{K}_{R[\underline{n}]}) \rightarrow K_n(R[\underline{n}][K])$  is on the level of spectra given by the map

$$E_{\phi^* \text{vCyc}}(K) \otimes_{\text{Or}(K)} \mathbf{K}_{R[\underline{n}]} \rightarrow K/K \otimes_{\text{Or}K} \mathbf{K}_{R[\underline{n}]} \simeq \mathbf{K}_{R[\underline{n}]}(K)$$

induced by  $E_{\phi^* \mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}}(K) \rightarrow K/K$ . The assumption in (i) is that this map of spectra is a weak equivalence. Using the fact that the realization of a map of simplicial spectra that is levelwise a weak equivalence is a weak equivalence and the identification

$$|E_{\phi^* \mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}}(K) \otimes_{\mathrm{Or}(K)} \mathbf{K}_{R[\bullet]}| \cong |E_{\mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}}(K) \otimes_{\mathrm{Or}(K)} \mathbf{K}_{R[\bullet]}|$$

we conclude that the (Fibered) Farrell-Jones Conjecture for  $(G, R[x_1, \dots, x_n])$  for all  $n$  implies the (Fibered)  $KH$ -Isomorphism Conjecture for  $(G, R)$  with the family of finite subgroups replaced by the family of virtually cyclic subgroups. By Remark 7.4 this is equivalent to the (Fibered)  $KH$ -Isomorphism Conjecture 7.3 for  $(G, R)$ .

Next we prove (ii) by reducing it to (i). For a group  $K$  we denote by  $p_K: K \times \mathbb{Z} \rightarrow K$  the canonical projection. We observe first that the (Fibered) Isomorphism Conjecture for  $(G, \mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}, R[\mathbb{Z}])$  is equivalent to the (Fibered) Isomorphism Conjecture  $(G \times \mathbb{Z}, (p_G)^* \mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}, R)$  because for every group  $K$  and every  $K$ -space  $X$  there is a natural isomorphism

$$H_n^K(X; \mathbf{K}_{R[\mathbb{Z}]}) \cong H_n^{K \times \mathbb{Z}}(p_K^* X; \mathbf{K}_R)$$

where  $p_K^* X$  denotes the  $K \times \mathbb{Z}$ -space obtained by restriction of  $X$  along  $p_K$  and because  $p_K^* E_{\mathcal{F}}(K) = E_{(p_K)^* \mathcal{F}}(K \times \mathbb{Z})$  for every family of subgroups of  $K$ . If the *Fibered* Isomorphism Conjecture holds for a family  $\mathcal{F}$ , then it will also hold for every family  $\mathcal{G}$  that contains  $\mathcal{F}$  [4, Lemma 1.6]. Because the family of virtually cyclic subgroups of  $G \times \mathbb{Z}$  is contained in  $(p_G)^* \mathcal{V}\mathcal{C}\mathcal{Y}\mathcal{C}$  the Fibered Farrell-Jones Conjecture for  $(G \times \mathbb{Z}, R)$  implies the Fibered Farrell-Jones Conjecture for  $(G, R[\mathbb{Z}])$ . By the Bass-Heller-Swan splittings [14] the latter is equivalent to the Fibered Farrell-Jones Conjecture for  $(G, R[x])$ . By induction on  $n$  this means that the assumption in (ii) implies the assumption of (i).  $\square$

**Remark 8.5.** It is not unreasonable to expect that the non-fibered version of Theorem 8.4 (ii) is also valid. Our argument would also prove the non-fibered version if we were to know that for every virtually cyclic group  $V$  the product  $V \times \mathbb{Z}$  satisfies the Farrell-Jones Conjecture. This seems very likely, but we could not find such a statement in the literature.

**Remark 8.6.** Let us briefly list some consequences of the Farrell-Jones Conjecture for algebraic  $K$ -theory. Suppose that the Farrell-Jones Conjecture for algebraic  $K$ -theory holds for the group  $G$  and every regular ring  $R$ . Now consider a group  $G$  and a regular ring  $R$  with the property that either  $\mathbb{Q} \subseteq R$  holds or  $G$  is torsionfree. The proof of Proposition 8.2, Remark 8.3 and Theorem 8.4 imply:

- (i) The  $KH$ -Isomorphism Conjecture 7.3 is true for  $G$  and  $R$ ;
- (ii) The canonical map  $K_n(RG) \rightarrow KH_n(RG)$  is bijective for  $n \in \mathbb{Z}$ ;

(iii)  $N^p K_n(RG) = 0$  for  $p \geq 1$  and  $n \in \mathbb{Z}$ , see [3, Proposition 7.4].

**Remark 8.7 (Injectivity of the  $KH$ -assembly map).** In many cases injectivity of the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

is proven by construction of a spectrum  $\mathbf{T}(R, G)$  and a map of spectra  $\mathbf{K}^{-\infty}RG \rightarrow \mathbf{T}(R, G)$  such that for many groups the composition of the assembly map on the level of spectra with this map is a weak equivalence. The construction of  $\mathbf{K}^{-\infty}RG \rightarrow \mathbf{T}(R, G)$  is always natural in the coefficient ring  $R$ . Therefore applying the arguments of the proof of Theorem 8.4 (i) we can use  $\mathbf{T}(R[\bullet], G)$  to split the  $KH$ -assembly map in this cases. This proves that the  $KH$ -assembly map is split injective for groups  $G$  of finite asymptotic dimension that admit a finite model for  $BG$  [2] and for groups  $G$  for which  $E_{\mathcal{FIN}}(G)$  has a compactification with certain properties [23].

## 9 Non-connective Waldhausen Nil

Before we can show that  $H_*^?(-; \mathbf{KH}_R)$  has the tree property, we will need to recall Waldhausen's work on  $K$ -theory of generalized free products [29]. We start with Waldhausen's Nil-groups.

**Definition 9.1 (Nil-categories).** Let  $R$  be a ring and  $X, Y, Z, W$  be  $R$ -bimodules.

- (i) The category  $\text{NIL}(R; X, Y)$  has as objects quadruples  $(P, Q, p, q)$ , where  $P$  and  $Q$  are finitely generated projective  $R$ -modules and  $p: P \rightarrow Q \otimes_R X$ ,  $q: Q \rightarrow P \otimes_R Y$  are  $R$ -linear maps subject to the following *nilpotence* condition: Let  $P_0 = Q_0 = 0$ ,  $P_{n+1} = p^{-1}(Q_n \otimes_R X)$  and  $Q_{n+1} = q^{-1}(P_n \otimes_R Y)$ . It is required that for sufficient large  $N$ ,  $P = P_N$  and  $Q = Q_N$ .
- (ii) The category  $\text{NIL}(R; X, Y, Z, W)$  has as objects quadruples  $(P, Q, p, q)$ , where  $P$  and  $Q$  are finitely generated projective  $R$ -modules and  $p: P \rightarrow Q \otimes_R X \oplus P \otimes_R Z$ ,  $q: Q \rightarrow P \otimes_R Y \oplus Q \otimes_R W$  are  $R$ -linear maps subject to the following *nilpotence* condition: Let  $P_0 = Q_0 = 0$ ,  $P_{n+1} = p^{-1}(Q_n \otimes_R X \oplus P_n \otimes_R Z)$  and  $Q_{n+1} = q^{-1}(P_n \otimes_R Y \oplus Q_n \otimes_R W)$ . It is required that for sufficient large  $N$ ,  $P = P_N$  and  $Q = Q_N$ .

Morphisms are in both cases  $R$ -linear maps  $P \rightarrow P'$ ,  $Q \rightarrow Q'$  that are compatible with  $p, p', q$  and  $q'$ . Both categories are exact categories, where sequences are exact whenever they map to exact sequences of modules under  $(P, Q, p, q) \mapsto P$  and  $(P, Q, p, q) \mapsto Q$ .

**Remark 9.2.** Let  $f_R: R \rightarrow S$  be a map of rings and  $f_X: X \rightarrow X'$ ,  $f_Y: Y \rightarrow Y'$  be maps over  $f_R$ , i.e.  $X'$  and  $Y'$  are  $S$ -bimodules,  $f_X(rxr') = f_R(r)f_X(x)f_R(r')$

and similar for  $f_Y$ . Then  $(f_R, f_X, f_Y)$  induce an exact functor  $\text{NIL}(R; X, Y) \rightarrow \text{NIL}(S; X', Y')$  sending  $(P, Q, p, q)$  to  $(P \otimes_R S, Q \otimes_R S, p_S, q_S)$  where  $p_S$  and  $q_S$  are the canonical maps. For example,  $p_S$  is the composition

$$P \otimes_R S \rightarrow Q \otimes_R X \otimes_R S \rightarrow Q \otimes_R X' \cong Q \otimes_R S \otimes_S X',$$

where the first map uses  $p$  and the second uses  $f_X$  and left multiplication of  $S$ . In particular, we get a functor  $S \otimes -: \text{NIL}(R; X, Y) \rightarrow \text{NIL}(S \otimes R; S \otimes X, S \otimes Y)$ . If  $f: S \rightarrow S'$  is a map of rings, then we get another functor  $f_*: \text{NIL}(S \otimes R; S \otimes X, S \otimes Y) \rightarrow \text{NIL}(S' \otimes R; S' \otimes X, S' \otimes Y)$ . The functoriality of  $\text{NIL}(R; X, Y, Z, W)$  is similar.

We review next [28, 2.5] in a slightly more modern language and discuss applications to Waldhausen Nil-categories.

A *sum ring* is a ring  $S$  together with elements  $v, \bar{v}, u$  and  $\bar{u}$  of  $S$  such that  $u\bar{u} = 1, v\bar{v} = 1$  and  $\bar{v}v + \bar{u}u = 1$ . This implies that  $u\bar{v} = 0$  and  $v\bar{u} = 0$ . Moreover, the map  $f_\oplus: S \oplus S \rightarrow S$  defined by  $(r, s) \mapsto \bar{u}ru + \bar{v}sv$  is a ring homomorphism. Let  $M, N$  be  $S$ -modules. Denote by  $(M, N)$  the direct sum  $M \oplus N$  considered as an  $S \oplus S$ -module. The  $S$ -modules  $(M, N) \otimes_{f_\oplus} S$  and  $M \oplus N$  (considered as an  $S$ -module as usual) are naturally isomorphic. Such an isomorphism and its inverse are given by

$$\begin{array}{llll} M \oplus N & \ni & m \oplus n & \mapsto & (m, n) \otimes (\bar{u} + \bar{v}) & \in & (M, N) \otimes_{f_\oplus} S \\ (M, N) \otimes_{f_\oplus} S & \ni & (m, n) \otimes r & \mapsto & mur \oplus nvr & \in & M \oplus N. \end{array}$$

An *infinite sum ring* is a sum ring together with a ring endomorphism  $f_\infty$  such that  $f_\oplus(r, f_\infty(r)) = f_\infty(r)$ .

**Remark 9.3.** The functor  $M \mapsto M \otimes_{f_\infty} S$  is an Eilenberg swindle on the category  $\mathcal{P}_S$  of finitely generated projective modules over such an infinite sum ring. Indeed,

$$\begin{aligned} M \otimes_{f_\infty} S &= M \otimes_{f_\oplus \circ (\text{id}_S, f_\infty)} S \\ &\cong M \otimes_{(\text{id}_S, f_\infty)} (S \oplus S) \otimes_{f_\oplus} S \\ &\cong (M, M \otimes_{f_\infty} S) \otimes_{f_\oplus} S \\ &\cong M \oplus (M \otimes_{f_\infty} S). \end{aligned}$$

The same swindle applies to Waldhausen's Nil-categories: Fix an infinite sum ring  $S$ . Let  $X$  and  $Y$  be bimodules over another ring  $R$ . Then the endofunctor  $(f_\infty)_*$  is equal to the composition

$$\begin{array}{c} \text{NIL}(S \otimes R; S \otimes X, S \otimes Y) \\ \downarrow (\text{id}, f_\infty)_* \\ \text{NIL}((S \oplus S) \otimes R; (S \oplus S) \otimes X, (S \oplus S) \otimes Y) \\ \downarrow (f_\oplus)_* \\ \text{NIL}(S \otimes R; S \otimes X, S \otimes Y) \end{array}$$

Using the natural isomorphism from above there is a natural transformation from this composition to  $\text{id} \oplus (f_\infty)_*$ . Thus,  $(f_\infty)_*$  is an Eilenberg swindle. This swindle is compatible with the two forgetful functors  $\text{NIL}(S \otimes R; S \otimes X, S \otimes Y) \rightarrow \mathcal{P}_{S \otimes R}$ . Analogous considerations apply to  $\text{NIL}(S \otimes R; S \otimes X, S \otimes Y, S \otimes Z, S \otimes Z, S \otimes W)$ .

The *cone ring*  $\Lambda\mathbb{Z}$  of  $\mathbb{Z}$  is the ring of column and row finite  $\mathbb{N} \times \mathbb{N}$ -matrices over  $\mathbb{Z}$ , i.e. matrices such that every column and every row contains only finitely many non-zero entries. The *suspension ring*  $\Sigma\mathbb{Z}$  is the quotient of  $\Lambda\mathbb{Z}$  by the ideal of finite matrices. For an arbitrary ring  $R$  we define  $\Lambda R = \Lambda\mathbb{Z} \otimes R$  and  $\Sigma R = \Sigma\mathbb{Z} \otimes R$ . We will view  $\Lambda$  and  $\Sigma$  as functors. Every bijection  $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  induces the structure of an infinite sum ring on the cone ring  $\Lambda R$ , cf. [28, p.355]. We can consider  $\Lambda R$  as a subring of the ring of column and row finite  $\mathbb{N} \times \mathbb{N}$ -matrices over  $R$ . However, this inclusion is not always an equality.

Next we want to define a non-connective spectrum associated to Waldhausen's Nil-categories. First we recall the construction for  $K$ -theory. Denote by  $\mathbb{K}R$  the  $K$ -theory space of a ring (obtained for example by applying Waldhausen's  $S_\bullet$ -construction to the category  $\mathcal{P}_R$  of finitely generated projective  $R$ -modules). The  $n$ -th space of the spectrum  $\mathbf{K}^{-\infty}R$  is by definition  $\mathbb{K}\Sigma^n R$ . The composition  $\mathbb{K}R \rightarrow \mathbb{K}\Lambda R \rightarrow \mathbb{K}\Sigma R$  is constant. The choice of an bijection  $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  gives an Eilenberg swindle on  $\Lambda R$ , cf. Remark 9.3. If we fix such a bijection we get a functorial way of contracting  $\mathbb{K}\Lambda R$  to the basepoint. This induces the structure maps  $\Sigma(\mathbb{K}\Sigma^n R) \rightarrow \mathbb{K}\Sigma^{n+1}R$ .

**Definition 9.4 (Non-connective Nil-spectra).** Let  $R$  be a ring and  $X$  and  $Y$  be  $R$ -bimodules. The (non-connective) spectrum  $\mathbf{NiL}^{-\infty}(R; X, Y)$  has  $\mathbb{K}\text{NIL}(\Sigma^n R; \Sigma^n X, \Sigma^n Y)$  as its  $n$ th space. Here  $\Sigma^n X = \Sigma^n \mathbb{Z} \otimes X$  and we define similarly  $\Sigma^n Y$ ,  $\Lambda X$  and  $\Lambda Y$ . The structure maps are defined in an analogous ways as for the non-connective  $K$ -theory spectrum: The functoriality discussed in Remark 9.2 allows us to consider the (constant) composition

$$\text{NIL}(C; A', B') \rightarrow \text{NIL}(\Lambda C; \Lambda A', \Lambda B') \rightarrow \text{NIL}(\Sigma C; \Sigma A', \Sigma B').$$

The structure maps for  $\mathbf{NiL}^{-\infty}(R; X, Y)$  are now defined using the Eilenberg swindle on the second category discussed in Remark 9.3. Similarly, we define a (non-connective) spectrum  $\mathbf{NiL}^{-\infty}(R; X, Y, Z, W)$  with

$$\mathbb{K}\text{NIL}(\Sigma^n R; \Sigma^n X, \Sigma^n Y, \Sigma^n Z, \Sigma^n W)$$

as its  $n$ th space.

An inclusion  $\alpha: C \rightarrow A$  of rings is called *pure* if  $A = \alpha(C) \oplus A'$  as  $C$ -bimodules. It is called *pure and free* if in addition  $A'$  is free as a left  $C$ -module. If  $H \rightarrow G$  is an inclusion of groups, then the inclusion  $RH \rightarrow RG$  of rings is pure and free. The following observation is straight forward.

**Lemma 9.5.** *If  $\alpha$  is pure (and free) then  $\Sigma\alpha$  and  $\Lambda\alpha$  are also pure (and free).*

Let  $\alpha: C \rightarrow A$  and  $\beta: C \rightarrow B$  be both pure. The ring  $R = A *_C B$ , the free product of  $A$  and  $B$ , amalgamated at  $C$  (w.r.t.  $\alpha, \beta$ ), is defined by the push-out

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \\ B & \longrightarrow & R. \end{array}$$

For group rings this corresponds to amalgamated products of groups.

**Lemma 9.6.** *The cone respectively suspension ring of  $A *_C B$  is naturally isomorphic to  $\Lambda A *_C \Lambda B$  respectively  $\Sigma A *_C \Sigma B$ .*

*Proof.* This follows from the universal property.  $\square$

Let  $\alpha, \beta: C \rightarrow A$  be pure and free. The Laurent extension w.r.t.  $\alpha$  and  $\beta$  is the universal ring  $R = A_{\alpha, \beta}\{t^{\pm 1}\}$  that contains  $A$  and an invertible element  $t$  and satisfies

$$\alpha(c)t = t\beta(c) \text{ for } c \in C.$$

Existence is explained in [29, p.149]. For group rings this corresponds to HNN-extensions.

**Lemma 9.7.** *The cone respectively suspension ring of  $A_{\alpha, \beta}\{t^{\pm 1}\}$  is naturally isomorphic to  $\Lambda A_{\Lambda\alpha, \Lambda\beta}\{T^{\pm 1}\}$  respectively  $\Sigma A_{\Sigma\alpha, \Sigma\beta}\{T^{\pm 1}\}$ .*

*Proof.* This follows from the universal property.  $\square$

## 10 Waldhausen's cartesian squares

Let  $\alpha: C \rightarrow A$  and  $\beta: C \rightarrow B$  be pure and free. Write  $A = \alpha(C) \oplus A'$  and  $B = \beta(C) \oplus B'$  as  $C$ -bimodules. Let  $R = A *_C B$ . Consider the square

$$\begin{array}{ccc} \text{NIL}(C; A', B') & \longrightarrow & \mathcal{P}_A \times \mathcal{P}_B \\ \downarrow & & \downarrow \\ \mathcal{P}_C & \longrightarrow & \mathcal{P}_R^* \end{array} \quad (10.1)$$

The two functors starting at the upper left hand corner are defined by sending  $(P, Q, p, q)$  to  $(P \oplus Q) \in \mathcal{P}_C$  respectively to  $(P \otimes_{\alpha} A, Q \otimes_{\beta} B)$ . The category  $\mathcal{P}_R^*$  is defined in [29, p.205]. It is a cofinal full subcategory of  $\mathcal{P}_R$  and contains all finitely generated free modules. There is an obvious natural transformation between the two ways to go through the diagram. However, there is also a not quite so obvious more complicated natural transformation that makes



use of  $p$  and  $q$ , cf. [29, 1.4, 11.3]: Let  $i_P: P \otimes_C A \otimes_A R \rightarrow P \otimes_C R$  and  $i_Q: Q \otimes_C B \otimes_B R \rightarrow Q \otimes_C R$  be the natural isomorphisms. Define  $N$  by the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & Q \otimes_C A \\ \downarrow & & \downarrow \\ P \otimes_C A \otimes_A R & \xrightarrow{N} & Q \otimes_C R \end{array}$$

and similarly  $M: Q \otimes_C B \otimes_B R \rightarrow P \otimes_C R$ . Then

$$\begin{pmatrix} i_P & M \\ N & i_Q \end{pmatrix}$$

is an isomorphism and defines the more complicated natural transformation. It is a result of Waldhausen [29, 11.3], that applying  $\mathbb{K}$  to (10.1) yields a homotopy cartesian square (w.r.t. the homotopy induced by the more complicated natural transformation). We will need a non-connective version of Waldhausen's result.

**Theorem 10.2. (Non-connective versions of Waldhausen's homotopy cartesian squares for amalgamation)** *We have the following diagram of spectra*

$$\begin{array}{ccc} \mathrm{NiL}^{-\infty}(C; A', B') & \longrightarrow & \mathbf{K}^{-\infty} A \wedge \mathbf{K}^{-\infty} B \\ \downarrow & & \downarrow \\ \mathbf{K}^{-\infty} C & \longrightarrow & \mathbf{K}^{-\infty} R \end{array}$$

*The more complicated natural transformations combine to a homotopy between the two ways to go through this diagram. The diagram is homotopy cartesian w.r.t. this homotopy.*

*Proof.* The diagram of spectra is obtained from (10.1) by tensoring everything in sight by  $\Sigma^n \mathbb{Z}$  (and applying  $\mathbb{K}$ ). We need to check compatibility with the structure maps. Those come from an intermediate diagram where we apply  $\Lambda \mathbb{Z} \otimes -$  and use an Eilenberg swindle. This Eilenberg swindle happens on the left of this tensor product, while everything else happens on the right. This proves compatibility with the structure maps. If we use  $\mathcal{P}_R^*$  rather than  $\mathcal{P}_R$  then the diagram is homotopy cartesian by Waldhausen's result and Lemma 9.6. However, since the former category contains all finitely generated free modules and we use non-connective  $K$ -theory we can also use  $\mathcal{P}_R$ .  $\square$

**Remark 10.3. (Waldhausen Nil for amalgamations vanishes for regular coherent rings)** Waldhausen proved that for a regular coherent ring  $C$ , the functor  $\mathcal{P}_C \times \mathcal{P}_C \rightarrow \mathrm{NiL}(C; A', B')$  defined by  $(P, Q) \mapsto (P, Q, 0, 0)$  induces an isomorphism in connective  $K$ -theory, [29, 12.2]. A priori, this

does not immediately imply that the induced map  $\alpha : \mathbf{K}^{-\infty}C \vee \mathbf{K}^{-\infty}C \rightarrow \mathbf{NiL}^{-\infty}(C; A', B')$  is a weak equivalence because it is not clear whether  $\Sigma C$  is again regular coherent. However, if  $C$  is regular or more generally, if  $C$  is a group ring with a regular coefficient ring over a regular coherent group, then  $\alpha$  is a weak equivalence. This can be seen as follows. The functor  $(P, Q, p, q) \mapsto (P, Q)$  splits  $\alpha$ , thus  $\alpha$  will be injective on homotopy groups. To prove surjectivity we use the fact that there is an in  $C$  natural map of rings  $C[\mathbb{Z}^n] \rightarrow \Sigma^n C$ , that is naturally split surjective in connective  $K$ -theory, [28, Section 6]. We get the following commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{C[\mathbb{Z}^n]} \times \mathcal{P}_{C[\mathbb{Z}^n]} & \longrightarrow & \mathbf{NiL}(C[\mathbb{Z}^n]; A'[\mathbb{Z}^n], B'[\mathbb{Z}^n]) \\ \downarrow & & \downarrow \\ \mathcal{P}_{\Sigma^n C} \times \mathcal{P}_{\Sigma^n C} & \longrightarrow & \mathbf{NiL}(\Sigma^n C; \Sigma^n A', \Sigma^n B') \end{array} \quad (10.4)$$

Apply  $C[\mathbb{Z}^n] \rightarrow \Sigma^n C$  map to the long exact sequence obtained from (10.1) by [29, 11.3]. A little diagram chase in the resulting ladder diagram shows that the right vertical map in (10.4) is surjective in connective  $K$ -theory. The assumptions on  $C$  imply that  $C[\mathbb{Z}^n]$  is regular coherent. Therefore the top horizontal map in (10.4) is an isomorphism in connective  $K$ -theory. Therefore the bottom horizontal map in (10.4) is also surjective in connective  $K$ -theory. This implies that  $\alpha$  is surjective on homotopy groups.

Next we discuss the analogous cartesian square for Laurent extensions. Let  $\alpha, \beta: C \rightarrow A$  be pure and free and  $R = A_{\alpha, \beta}\{t^{\pm 1}\}$ . We denote by  $\iota: A \rightarrow R$  the inclusion. Write  $A = \alpha(C) \oplus A'$  and  $A = \beta(C) \oplus A''$  as  $C$ -bimodules. Consider the square

$$\begin{array}{ccc} \mathbf{NiL}(C;_{\alpha} A'_{\alpha, \beta} A''_{\beta, \beta} A_{\alpha, \alpha} A_{\beta}) & \longrightarrow & \mathcal{P}_A \\ \downarrow & & \downarrow \iota_* \\ \mathcal{P}_C & \xrightarrow{(\iota \circ \alpha)_*} & \mathcal{P}_R^* \end{array} \quad (10.5)$$

Here we use  $\alpha$  and  $\beta$  to indicate the  $C$ -bimodule structures. The two functors starting at the upper right hand corner are defined by sending  $(P, Q, p, q)$  to  $(P \oplus Q) \in \mathcal{P}_C$  respectively to  $(P \otimes_{\alpha} A \oplus Q \otimes_{\beta} A)$ . The category  $\mathcal{P}_R^*$  is defined in [29, p.205]. It is a cofinal full subcategory of  $\mathcal{P}_R$  and contains all finitely generated free modules. As before there is an obvious and a more complicated natural transformation between the two ways to go through the diagram [29, 2.4, 12.3]: Let  $i_P: P \otimes_{\alpha} A \otimes_A \otimes R \rightarrow P \otimes_{\alpha} R$  and  $i_Q: Q \otimes_{\beta} A \otimes R \rightarrow Q \otimes_{\beta} R$  denote the canonical isomorphisms. (Here  $i_Q$  uses an extra  $t$ , i.e.  $i_Q(y \otimes a \otimes r) = y \otimes tar$ .) These isomorphisms give the obvious natural transformation. The more complicated natural transformation is obtained by adding a nilpotent term which we review next. Write  $p = p_0 + p_1$ , where  $p_0: P \rightarrow P \otimes_{\beta} A_{\alpha}$  and  $p_1: P \rightarrow Q \otimes_{\alpha} A'_{\alpha}$ . Define  $N_0$  and  $N_1$  by the

commutative diagrams

$$\begin{array}{ccc}
P & \xrightarrow{p_0} & P \otimes_{\beta} A_{\alpha} \\
\downarrow & & \downarrow \\
P \otimes_{\alpha} A \otimes_A R & \xrightarrow{N_0} & P \otimes_{\alpha} R
\end{array}
\quad
\begin{array}{ccc}
P & \xrightarrow{p_1} & Q \otimes_{\beta} A'_{\alpha} \\
\downarrow & & \downarrow \\
P \otimes_{\alpha} A \otimes_A R & \xrightarrow{N_1} & Q \otimes_{\alpha} R
\end{array}$$

(The second vertical arrow is  $x \otimes a \mapsto x \otimes ta$ .) Write  $q = q_0 + q_1$ , where  $q_0: Q \rightarrow Q \otimes_{\alpha} A_{\beta}$  and  $q_1: Q \rightarrow P \otimes_{\beta} A''_{\beta}$ . Define  $M_0$  and  $M_1$  by the commutative diagrams

$$\begin{array}{ccc}
Q & \xrightarrow{q_0} & Q \otimes_{\alpha} A_{\beta} \\
\downarrow & & \downarrow \\
Q \otimes_{\beta} A \otimes_A R & \xrightarrow{M_0} & Q \otimes_{\alpha} R
\end{array}
\quad
\begin{array}{ccc}
Q & \xrightarrow{q_1} & P \otimes_{\alpha} A''_{\beta} \\
\downarrow & & \downarrow \\
Q \otimes_{\beta} A \otimes_A R & \xrightarrow{M_1} & P \otimes_{\alpha} R
\end{array}$$

(The forth vertical arrow is  $x \otimes a \mapsto x \otimes ta$ .) The more complicated natural transformation is then given by the isomorphism

$$\begin{pmatrix} i_P & 0 \\ 0 & i_Q \end{pmatrix} + \begin{pmatrix} N_0 & M_1 \\ N_1 & M_0 \end{pmatrix}.$$

It is a result of Waldhausen [29, 12.3], that applying  $\mathbb{K}$  to (10.5) yields a homotopy cartesian square (w.r.t. the homotopy induced by the more complicated natural transformation). The arguments used to prove Theorem 10.2 can be used to prove a non-connective version of this result.

**Theorem 10.6. (Non-connective versions of Waldhausen's homotopy cartesian squares for Laurent extensions)** *We have the following diagram of spectra*

$$\begin{array}{ccc}
\mathbf{Nil}^{-\infty}(C;_{\alpha} A'_{\alpha,\beta} A''_{\beta,\beta} A_{\alpha,\alpha} A_{\beta}) & \longrightarrow & \mathbf{K}^{-\infty}(A) \\
\downarrow & & \downarrow \mathbf{K}^{-\infty}(\iota) \\
\mathbf{K}^{-\infty}(C) & \xrightarrow{\mathbf{K}(\iota \circ \alpha)} & \mathbf{K}^{-\infty}(R)
\end{array}$$

The more complicated natural transformations combine to a homotopy between the two ways to go through this diagram. The diagram is homotopy cartesian w.r.t. this homotopy.

**Remark 10.7. (Waldhausen Nil for Laurent extensions vanishes for regular coherent rings)** The reasoning in Remark 10.3 also applies to  $\mathbf{Nil}(C;_{\alpha} A'_{\alpha,\beta} A''_{\beta,\beta} A_{\alpha,\alpha} A_{\beta})$ . If  $C$  is a group ring with a regular coefficient ring over a regular coherent group, then the functor  $(P, Q) \mapsto (P, Q, 0, 0)$  induces a weak equivalence  $\mathbf{K}^{-\infty} C \vee \mathbf{K}^{-\infty} C \rightarrow \mathbf{Nil}^{-\infty}(C;_{\alpha} A'_{\alpha,\beta} A''_{\beta,\beta} A_{\alpha,\alpha} A_{\beta})$ .

## 11 The tree property for Homotopy K-Theory

This section contains the proof of the following result.

**Theorem 11.1 (Continuity and tree-property for  $H_*^?(-; \mathbf{KH}_R)$ ).** *The equivariant homology theory  $H_*^?(-; \mathbf{KH}_R)$  is continuous and has the tree property.*

Let  $X, Y, Z$  and  $W$  be bimodules over  $R$ . Consider the simplicial spectra  $\underline{n} \mapsto \mathbf{NIL}(\mathbb{Z}[\underline{n}] \otimes R; \mathbb{Z}[\underline{n}] \otimes X, \mathbb{Z}[\underline{n}] \otimes Y)$  and  $\underline{n} \mapsto \mathbf{NIL}(\mathbb{Z}[\underline{n}] \otimes R; \mathbb{Z}[\underline{n}] \otimes X, \mathbb{Z}[\underline{n}] \otimes Y, \mathbb{Z}[\underline{n}] \otimes Z, \mathbb{Z}[\underline{n}] \otimes W)$ . We will denote the realization of these simplicial spectra by  $\mathbf{NH}(R; X, Y)$  or  $\mathbf{NH}(R; X, Y, Z, W)$  respectively. However, the point here is that this process kills the additional information in Waldhausen's Nil-groups.

**Proposition 11.2.**

- (i) *The functor  $\mathcal{P}_R \times \mathcal{P}_R \rightarrow \mathbf{NIL}(R; X, Y)$  defined by  $(P, Q) \mapsto (P, Q, 0, 0)$  induces an equivalence  $\mathbf{KH}(R) \vee \mathbf{KH}(R) \rightarrow \mathbf{NH}(R; X, Y)$ .*
- (ii) *The functor  $\mathcal{P}_R \times \mathcal{P}_R \rightarrow \mathbf{NIL}(R; X, Y, Z, W)$  defined by  $(P, Q) \mapsto (P, Q, 0, 0)$  induces an equivalence  $\mathbf{KH}(R) \vee \mathbf{KH}(R) \rightarrow \mathbf{NH}(R; X, Y, Z, W)$ .*

*Proof.* We prove only (i), the proof of (ii) is similar. It suffices to show that the functor  $(P, Q, p, q) \mapsto (P, Q, 0, 0)$  mapping the simplicial category  $\mathbf{N}(\bullet) = \mathbf{NIL}(\mathbb{Z}[\bullet] \otimes R; \mathbb{Z}[\bullet] \otimes X, \mathbb{Z}[\bullet] \otimes Y)$  to itself is simplicially homotopic to the identity. Such a homotopy  $\mathbf{N}(\bullet) \times \Delta_\bullet^1 \rightarrow \mathbf{N}(\bullet)$  is given by

$$(P, Q, p, q) \mapsto (P, Q, \sum_{j \in f^{-1}(0)} t_j \otimes p, \sum_{j \in f^{-1}(0)} t_j \otimes q)$$

where  $f: \underline{n} \mapsto \underline{1}$ . □

The homotopy algebraic  $K$ -theory of a free product or a Laurent extension does therefore not involve Nil-groups.

**Theorem 11.3. (Homotopy cartesian squares for Homotopy  $K$ -theory)**

- (i) *Consider the free product  $R = A *_C B$  (w.r.t. pure and free maps  $\alpha: C \rightarrow A$  and  $\beta: C \rightarrow B$ ). Then the commutative diagram*

$$\begin{array}{ccc} \mathbf{KH}(C) \vee \mathbf{KH}(C) & \xrightarrow{\mathbf{KH}(\alpha) \vee \mathbf{KH}(\beta)} & \mathbf{KH}(A) \vee \mathbf{KH}(B) \\ \text{id} \vee \text{id} \downarrow & & \downarrow \mathbf{KH}(\iota_A) \vee \mathbf{KH}(\iota_B) \\ \mathbf{KH}(C) & \xrightarrow{\mathbf{KH}(\alpha \circ \iota_A)} & \mathbf{KH}(R) \end{array}$$

*is homotopy cartesian. Here  $\iota_A$  and  $\iota_B$  are the obvious inclusions of rings.*

(ii) Consider the Laurent extension  $R = A_{\alpha,\beta}\{t^{\pm 1}\}$  (w.r.t. pure and free maps  $\alpha: C \rightarrow A$  and  $\beta: C \rightarrow A$ ). Then the diagram

$$\begin{array}{ccc} \mathbf{KH}(C) \vee \mathbf{KH}(C) & \xrightarrow{\mathbf{KH}(\alpha) \vee \mathbf{KH}(\beta)} & \mathbf{KH}(A) \\ \text{id} \vee \text{id} \downarrow & & \downarrow \mathbf{KH}(\iota) \\ \mathbf{KH}(C) & \xrightarrow{\mathbf{KH}(\iota \circ \alpha)} & \mathbf{KH}(R) \end{array}$$

is homotopy cartesian w.r.t. the obvious homotopy between the two ways to go through this diagram, cf. Section 10. Here  $\iota$  is the obvious inclusions of rings.

*Proof.* The realization of a simplicial diagram of spectra that is degreewise homotopy cartesian is again homotopy cartesian. Thus the result follows by combining Theorems 10.2 or 10.6 respectively and Proposition 11.2. There is no longer a difference between the obvious and the more complicated natural transformation, since we got rid of the nil-categories.  $\square$

*Proof of Theorem 11.1.* We use Lemma 5.1. We discuss first continuity, i.e. condition 5.1 (i). For  $\mathbf{K}_R$  this follows from the compatibility of  $K$ -theory with directed colimits [22]. Since realizations of simplicial spectra commutes with directed colimits, this implies condition 5.1 (i).

Next we discuss the tree property, i.e. conditions 5.1 (ii) and (iii). Note that  $\mathbf{KH}_R$  is a 2-functor as discussed in Remark 5.6. For  $\mathbf{K}_R$  this holds since natural equivalences between functors  $F$  and  $G$  induce naturally a homotopy from  $\mathbf{K}^{-\infty}(F)$  to  $\mathbf{K}^{-\infty}(G)$ . This homotopy is preserved under realization. Now observe that the obvious homotopy in Theorem 11.3 (ii) is in the case of an HNN-extension the homotopy coming from conjugation as discussed in Remark 5.6 (ii). Thus conditions 5.1 (ii) and (iii) are satisfied for  $\mathbf{KH}_R$  by Theorem 11.3 and Remark 5.6.  $\square$

Using Theorems 10.2 and 10.6 and Remarks 10.3 and 10.7 the above arguments also prove a version of Theorem 11.1 for algebraic  $K$ -theory.

**Theorem 11.4 (Continuity and tree-property for  $H_*^?(-; \mathbf{K}_R)$ ).** *The equivariant homology theory  $H_*^?(-; \mathbf{K}_R)$  is continuous and if  $R$  is regular then it has the regular tree property.*

Now we can finish the proof of the various results stated in the Introduction. We start with Theorem 0.5 and Theorem 0.11. The property (FIN) respectively (VCYC) hold for trivial reasons. Similar (SUB) in the Fibered case is a formal consequence of the Definitions, compare Lemma 1.2. The property (COL) is a consequence of Proposition 3.4, Theorem 11.1 and Theorem 11.4. The properties (TREE) respectively (TREE $_{\mathcal{R}}$ ) follow from Theorem 4.2 and Theorem 11.1 respectively Theorem 11.4. Theorem 0.7 is a

consequence of Corollary 4.4 and Theorem 11.1. Theorem 0.8 is a consequence of Theorem 0.5, Proposition 8.2 and Remark 8.3. It remains to prove Proposition 0.9.

*Proof.* (i) This result is stated in [19, page 133]. We give an outline of the proof. We consider first a group  $G$  which possesses a finite presentation with one relation. Let  $r$  be the number of generators appearing in the word describing the relation. If  $r \leq 1$ , then  $G$  is the amalgamated product of a free group and a finite cyclic group. Obviously any finite group and  $\mathbb{Z}$  belong to  $\mathcal{C}_0$  and  $\mathcal{C}_0$  is closed under free products. Hence  $G$  belongs to  $\mathcal{C}_0$ . It remains to treat  $r \geq 2$ . Here we use induction over the length  $l$  of the word describing the relation. In our case  $l \geq 2$ . Then  $G$  acts on a tree with stabilizers which are subgroups of one-relator groups whose relation have length  $\leq (l-1)$ , [8, Theorem 7.7] and hence belong to  $\mathcal{C}_0$  by the induction hypothesis. Therefore  $G$  belongs to  $\mathcal{C}_0$ . For a general one-relator group  $G$  there are finitely generated subgroups  $G_i$  which are free or one-relator groups, such that  $G$  is the directed colimit over the  $G_i$ .

(ii) Let  $\{1\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$  be a sequence of subgroups such that  $G_{i-1}$  is normal in  $G_i$  and the quotient  $G_i/G_{i-1}$  is free for  $i = 1, 2, \dots, n$ . We prove by induction over  $n$  that  $G$  belongs to  $\mathcal{C}_0$ . The induction beginning  $n = 0$  is trivial because of property (FIN), the induction step done as follows. We can write  $G_n/G_{n-1}$  as a directed union of its finitely generated subgroups. Hence  $G_n$  is the directed union of the preimages of the finitely generated subgroups of  $G_n/G_{n-1}$ . Since any finitely generated subgroup of a free group  $F$  is a finitely generated free group, it suffices to treat the case, where  $G_n/G_{n-1}$  is finitely generated free by property (COL). Since  $G_n/G_{n-1}$  acts on a tree with trivial stabilizers,  $G_n$  acts on a tree with stabilizers which are all isomorphic to  $G_{n-1}$  and hence belong to  $\mathcal{C}_0$ . Hence  $G$  belongs to  $\mathcal{C}_0$  by property (TREE).

(iii), (iv), and (v) These follow from [29, Theorem 17.5 on page 250].  $\square$

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